m2pm3l8.tex
Lecture 8. 28.1.2010.

6. *The Theorems of Bolzano &Weierstrass and Cantor.* We quote (proofs not examinable):

Theorem (Bolzano-Weierstrass). If S is an infinite bounded set in $\mathbf{R}^d$ (or $\mathbf{C}$) then S has at least one limit point.

Theorem (Cantor; Nested Sets Theorem). If K_n is a decreasing sequence of closed and bounded sets in $\mathbf{R}^d$ or $\mathbf{C}$, i.e.

$$K_1 \supset K_2 \supset \ldots \supset K_n \supset \ldots$$

then their intersection $\cap_{n=1}^{\infty} K_n$ is non-empty.

The proofs (see the Handout) use *repeated bisection.*

Note that the condition that the K_n be bounded is essential here. For in $\mathbf{R}$, the sets $[n, \infty)$ are decreasing and bounded, but their intersection is empty. (One can think of their intersection as 'the point at $+\infty$', but this is *not* a real number, so not in $\mathbf{R}$. It is in the *extended* real line $\mathbf{R}^*$, which unlike the real line is 'compact', in a sense to which we now turn.)

7. *Compactness.* We usually write: open sets as G (G for geöffnet = open, German), closed sets as F (F for fermé = closed, French), $\mathcal{G}$, $\mathcal{F}$ for the classes of open sets and of closed sets.

Defn. A collection $\{G_\alpha : G_\alpha \in \mathcal{G}, \alpha \in A\}$ (G_α open, A some index set) is an *open covering* for S if $S \subset \cup_{\alpha \in A} G_\alpha$ ("the G_α covers S").

We quote that in $\mathbf{R}^d$, or $\mathbf{C}$, one can always reduce an (uncountably infinite) open covering to a *finite or countably infinite* subcovering (i.e., some finite or countably infinite subfamily of the G_α still covers S). (This is because in $\mathbf{R}$, each real is a limit of a sequence of rationals. One says that the rationals (which are countable) are *dense* in the reals, and that the reals, having a countable dense set, are *separable*. Similarly for $\mathbf{R}^d$, $\mathbf{C}$.)

For some sets S, one can always reduce to a *finite* subcovering.

Defn. A set S is *compact* if any open covering of S contains a finite subcovering.

We usually write compact sets as K (K for kompakt=compact, German), $\mathcal{K}$

for the class of compact sets.

We quote: in any metric space (e.g. $\mathbf{R}^d$ or $\mathbf{C}$):
(i) S compact implies S closed.
(ii) S compact implies S bounded.
Combining: in a metric space, S compact implies S closed and bounded. The converse is harder, and needs restriction. We quote:

Theorem (Heine-Borel). In Euclidean space $\mathbf{R}^d$, or $\mathbf{C}$, S compact iff S closed and bounded.
Examples.
1. *Hilbert space* ℓ^2: $\ell^2 := \{x = (x_n)_1^\infty : \|x\| = \sqrt{\sum_{n=1}^\infty |x_n|^2} < \infty\}$. This is a metric space, under the normal Euclidean distance. The unit ball $B_1 := \{x : \|x\| \leq 1\}$ is closed and bounded. But B is not compact. For example, the unit vectors $\delta_n = (\delta_{nm})_{m=1}^\infty$ (Kronecker delta), are all $\sqrt{2}$ apart (Pythagoras' Theorem). So no subsequence can converge. This says that B is not 'sequentially compact', and (we quote) in a metric space sequential compactness is the same as compactness. So B is not compact, as B is a metric space. (So the Heine-Borel Theorem depends on Euclidean space being *finite-dimensional*).
2. The complex plane $\mathbf{C}$ is not compact.
First Proof.
We use the Argand representation, and work in $\mathbf{R}^2$. Then $\mathbf{C}$ is not bounded (though it is closed), so $\mathbf{C}$ is not compact, by Heine-Borel.
Second Proof
We use stereographic projection and work in $\mathbf{R}^3$. Then $\mathbf{C} \leftrightarrow \Sigma'$ (punctured sphere) - bounded but not closed. So $\mathbf{C}$ is not compact, by Heine-Borel.
3. The extended complex plane $\mathbf{C}^*$ is compact.
Proof. By stereographic projection, $\mathbf{C}^* \leftrightarrow \Sigma$, closed and bounded, so compact by Heine-Borel.
No 'second proof': $\mathbf{C}^*$ is not Euclidean.

Theorem (Heine). If f is a continuous function on a compact set S, f is uniformly continuous on S.
Cor. (Heine). If $f : [a, b] \to \mathbf{R}$ is continuous, f is uniformly continuous on $[a, b]$.
See Handout (not examinable).