m2pm3l9.tex Lecture 9. 29.1.2010.

8. Cauchy's General Principle of Convergence. In \mathbf{R}^d or \mathbf{C} , a sequence (x_n) is called Cauchy if

 $\forall \epsilon > 0, \exists N \text{ s.t. } \forall m, n \ge N, |x_m - x_n| < \epsilon.$

It is called *convergent* if

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall n \ge N, |x_n - x| < \epsilon.$$

Convergent \Rightarrow Cauchy: For then, by the Triangle Inequality,

$$\forall m, n \ge N, |x_m - x_n| \le |x_m - x| + |x - x_n| < \epsilon + \epsilon = 2\epsilon.$$

So (x_n) is Cauchy. //

The converse, Cauchy \Rightarrow convergent, is *true* in \mathbf{R}^d (or \mathbf{C}), but *false* in \mathbf{Q} (or \mathbf{Q}^d). E.g. The sequence of decimal approximations to $\sqrt{2}$, 1, 1.4, 1.414, ... is convergent to $\sqrt{2}$ in \mathbf{R} , but not convergent in \mathbf{Q} (as $\sqrt{2} \notin \mathbf{Q}$).

Defn. Call a metric space complete if all Cauchy sequences are convergent.

We quote: we can embed any metric space in a larger metric space, its *completion*, s.t.: (i) the new space is complete; (ii) the original space is *dense* in the new space (any point in the new space is a limit of a convergent sequence of points in the old space). The prototype of this is completing the rationals to get the reals (Cantor, 1872). The general procedure (which came much later, with metric spaces) is essentially no more complicated.

9. Upper and lower limits. The sequence $+1, -1, +1, -1, ..., (-1)^{2n}, (-1)^{2n+1}, ...$ is not convergent. But it contains convergent subsequences, and has 'upper limit +1, and lower limit -1'. Any sequence (x_n) of reals has an upper and lower limit, limsup and liminf, (possibly $\pm \infty$):

$$\limsup_{n \to \infty} x_n = \lim_{n \to \infty} (\sup_{k > n} x_k), \qquad \liminf_{n \to \infty} x_n = \lim_{n \to \infty} (\inf_{k \ge n} x_k).$$

10. O and o. We write 'f(x) = O(g(x))' (as $x \to \infty$, say) to mean 'f(x)/g(x) is bounded', 'f(x) = o(g(x))' to mean ' $f(x)/g(x) \to 0$. Thus f = O(g) means

'f is of the order of g, while f = o(g) means 'f is of smaller order than g'. We shall make heavy use of this very useful notation in Chapter III.

11. Power Series. Recall two convergence tests for $\sum_{0}^{\infty} a_n$.

Ratio Test (D'Alembert). If $|a_{n+1}/a_n| \to \ell$ as $n \to \infty$, $\sum a_n$ is (i) convergent if $\ell < 1$; (ii) divergent if $\ell > 1$. The test is inconclusive if $\ell = 1$.

Root Test (Cauchy). Write $\rho := \limsup(|a_n|^{1/n})$ (i) If $\rho < 1$, $\sum |a_n|$ is convergent ($\sum a_n$ is absolutely convergent), (ii) If $\rho > 1$, $\sum |a_n|$ is divergent. The test is inconclusive if $\rho = 1$.

The Root Test is more general: if $a_n > 0$, $a_{n+1}/a_n \to \ell \Rightarrow a_n^{1/n} \to \ell$ also. Usually we use the Ratio Test for a_n involving factorials n!, and use the Root test for a_n involving nth powers, x^n .

Defn. A power series in $z \in \mathbf{C}$ is a series of the form $\sum_{n=0}^{\infty} a_n z^n$, $(a_n \in \mathbf{C})$. The series may converge for: all z (e.g. the exponential series $e^z = \sum_0^{\infty} z^n/n!$); some but not all z (e.g. the geometric series 1/(1-z)); only for z = 0, to a_0 (e.g. $\sum_0^{\infty} n! z^n$) – a trivial case, omitted. We write $\rho := \limsup |a_n|^{1/n}$. By the Root Test for $\sum a_n z^n$: If $\rho |z| < 1$, i.e. $|z| < 1/\rho$, $\sum a_n z^n$ converges (absolutely). If $\rho |z| > 1$, i.e. $|z| > 1/\rho$, $\sum a_n z^n$ diverges. If $\rho |z| = 1$, i.e. $|z| = 1/\rho$, the test is inconclusive.

Defn. Write $R := 1/\rho = 1/\limsup |a_n|^{1/n}$: $\sum a_n z^n$ is absolutely convergent for |z| < R, and divergent for |z| > R. We call R the Radius of Convergence (R of C) of $\sum a_n z^n$, |z| = R its Circle of Convergence. Similarly for $\sum a_n (z - z_0)^n$, with base-point z_0 .

So a power series: converges (absolutely) *inside* its circle of convergence; diverges *outside* its circle of convergence; may do either *on* its circle of convergence. $\sum a_n z^n$ converges (absolutely and) uniformly in $|z| \leq R_1, R_1 < R$: A power series converges (absolutely and) uniformly in *closed discs inside* the circle of convergence.