M2PM3 PROBLEMS 1. 15.1.2010

Q1. (i) Show that a real number x is rational iff its decimal expansion terminates or recurs.
(ii) What can be said about the decimal expansion of m/n (cancelled down to its lowest terms)?
(iii) What about binary expansions? ternary? etc.

Q2. For $f(x) := \exp(-1/x^2)$,
(i) Show (by induction or otherwise) that

$$f^{(n)}(x) = P_n(1/x)\exp(-1/x^2)$$

with $P^{(n)}$ a polynomial.
(ii) Deduce that $f^{(n)}(0) = 0$ for all n.
(iii) Deduce that the Taylor expansion of f about 0 converges for all x, but to 0 and not to $f(x)$.

Q3. *Spherical polar coordinates.* In spherical polar coordinates (r, θ, ϕ), parametrize the unit sphere $r = 1$ by (θ, ϕ) (longitude, colatitude).

Where does this coordinate representation fail to be unique, and why?

Q4. For $f_n(x) := nx/(1 + n^2x^2)$ $(x \in [0, \infty),\ n = 1, 2, \ldots)$:
(i) Show that $f_n(x) \to 0$ as $n \to \infty$, for all $x \geq 0$, i.e. $f_n \to 0$ *pointwise* on $[0, \infty)$.
(ii) Show that $\sup_{x\in[0,\infty)} f_n(x)$ does not tend to 0.
(iii) Deduce that f_n does not tend to 0 *uniformly* on $[0, \infty)$.

Q5. Show from first principles that $\cos(\pi/3) = 1/2$, $\sin(\pi/3) = \sqrt{3}/2$.

NHB