M2PM3 PROBLEMS 3, 4.2.2010

Q1 (*Lagrange's identity*). Show that for $z_1, \dots, z_n$, $w_1, \dots, w_n$ complex,

$$\sum_1^n |z_i|^2 \sum_1^n |w_j|^2 - |\sum_1^n z_i w_i|^2 = \sum_{1 \le i < j \le n} |z_i \bar{w}_j - z_j \bar{w}_i|^2.$$

Deduce the Cauchy-Schwarz inequality

$$|\sum_1^n z_i w_i| \le \sqrt{\sum_1^n |z_i|^2} \sqrt{\sum_1^n |w_j|^2}.$$

Q2 (*Weierstrass t-substitution*). If $t := \tan \frac{1}{2}\theta$, show that

$$d\theta = \frac{2dt}{1+t^2}, \qquad \sin\theta = \frac{2t}{1+t^2}, \qquad \cos\theta = \frac{1-t^2}{1+t^2}, \qquad \tan\theta = \frac{2t}{1-t^2}.$$

Show that for $-1 < c < 1$,

$$\int_0^\pi \frac{d\theta}{1 + c\cos\theta} = \frac{\pi}{\sqrt{1-c^2}}.$$

(We will meet this example early in Ch. III using complex methods – residue calculus).

Q3 (*Bessel functions of integer order*). For z, t complex, define $J_n(z)$ by its *generating function*:

$$\exp\left(\frac{1}{2}z(t - \frac{1}{t})\right) = \sum_{n=-\infty}^{\infty} t^n J_n(z)$$

(that is, the $J_n(z)$ are defined as the expansion coefficients of the function on the left). Show that

$$J_n(z) = \sum_{j=0}^{\infty} \frac{(-)^j (\frac{1}{2}z)^{n+2j}}{j!(n+j)!}.$$

Note: (i) This formula also holds for $J_\nu(z)$, the Bessel function of arbitrary order ν, which you may have met.
(ii) We shall see later how to derive other properties of $J_n(z)$ from the generating function, such as integral representations.
(iii) We shall meet this kind of 'two-sided power series' later as Laurent series in Ch. II.

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