M2PM3 PROBLEMS 5. 18.2.2010

Q1. For each of the following functions $u = u(x, y)$, find the $v = v(x, y)$ so that $f := u + iv$, $f = f(z)$ is holomorphic.
(i) $u = x^2 - y^2 - x$.
(ii) $u = x - y/(x^2 + y^2)$.

Q2. Show that as θ increases from 0 to $\pi/2$, $\sin\theta/\theta$ decreases from 1 to $2/\pi$. [We shall need this result from Real Analysis in Chapter III: Applications.]

Q3. Let $z_0 := e^{i\alpha}$, α not an integer multiple of π,

$$f(z) := \frac{z^n + z^{-n} - z_0^n - z_0^{-n}}{(z - z_0)(z - z_0^{-1})}.$$

Show that f is holomorphic for all non-zero z (including z_0 and z_0^{-1}), while near 0 $f(z)$ can be expanded in positive and negative powers of z, the coefficient of z^{-1} being

$$\frac{(1 + z_0^2 + \ldots + z_0^{2(n-1)})}{z_0^{n-1}} = \frac{\sin n\alpha}{\sin\alpha}.$$

[If f is not holomorphic at z_0, and we expand f about z_0 in positive and negative powers of $z - z_0$, the coefficient of $(z - z_0)^{-1}$ is called the *residue* of f at z_0, and plays a dominant role in Ch. III.]

Q4. We know from Problems 3 that the Gamma function $\Gamma(z)$ is holomorphic except at the points $z = 0, -1, -2, \ldots$.
(i) By induction or otherwise, show that for complex ζ,

$$\Gamma(-n + \zeta) \sim \frac{(-)^n}{n!\zeta} \qquad (\zeta \to 0, n = 0, 1, 2, \ldots),$$

$$\Gamma(-n + \zeta)\Gamma(n + 1 - \zeta) \sim (-)^n/\zeta \qquad (\zeta \to 0, n = 0, \pm 1, \pm 2, \ldots),$$

and so also that

$$\Gamma(n + \zeta)\Gamma(1 - n - \zeta) \sim (-)^n/\zeta \qquad (\zeta \to 0, n = 0, \pm 1, \pm 2, \ldots).$$

(ii) Show that for n an integer and $z = n + \zeta$,

$$\pi/\sin\pi z \sim (-)^n/\zeta \qquad (\zeta \to 0, n = 0, \pm 1, \pm 2, \ldots).$$

[We shall see later that $\Gamma(z)\Gamma(1 - z) = \pi/\sin\pi z$, so this similarity is not accidental!]

NHB