M2PM3 PROBLEMS 6. 4.3.2010

Q1 (*Liouville's theorem on* $\mathbf{C}^*$). We say that $f(z)$ has a property *at infinity* if $f(1/z)$ has the property at 0. Show that if f is holomorphic in the extended complex plane $\mathbf{C}^*$ (i.e., entire – holomorphic in $\mathbf{C}$ – and holomorphic at ∞), f is constant.

So a non-constant entire function has a singularity at ∞. Give some examples.

Q2. By considering $\int_\gamma dz/z$ with γ the ellipse $x^2/a^2 + y^2/b^2 = 1$ ($a,\ b > 0$) parametrized by $x = a\cos\theta$, $y = b\sin\theta$ and using CIF, or otherwise, show that

$$\int_0^{2\pi} \frac{d\theta}{a^2\cos^2\theta + b^2\sin^2\theta} = \frac{2\pi}{ab}.$$

Q3. With $\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}dt$, show (by Real Analysis) that

$$\Gamma(x)\Gamma(1-x) = \int_0^\infty \frac{v^{x-1}}{1+v}dv \qquad (0 < x < 1).$$

Q4. Obtain

$$\int_0^\infty \frac{\sin x}{x}dx = \frac{1}{2}\pi$$

by real methods, as follows. Write

$$F(t) := \int_0^\infty e^{-xt}\frac{\sin x}{x}dx \qquad (t \geq 0).$$

Assuming that one can 'differentiate under the integral sign' (one can – you may assume this), obtain

$$F'(t) = -\int_0^\infty e^{-xt}\sin x dx \qquad (t > 0).$$

Integrate by parts twice to show that

$$F'(t) = -1/(1+t^2).$$

Integrate to find $F(t)$, and use $F(t) \to 0$ as $t \to \infty$ to deduce $I := F(0+) = \pi/2$.

Q5. If f is entire and $f(z) = O(|z|^k)$ as $|z| \to \infty$, show that f is a polynomial of degree $\leq k$.

NHB