M2PM3 PROBLEMS 8, 18.3.2010

Q1 (*Wallis' product for* π: *Real Analysis*). By integrating by parts, or otherwise, show that if $I_n := \int \sin^n x \, dx$,

$$nI_n = -\sin^{n-1} x \cos\, x + (n-1)I_{n-2}.$$

Deduce that if $J_n := \int_0^{\pi/2} \sin^n x \, dx$, one has the *reduction formula*

$$J_n = \frac{n-1}{n}.J_{n-2}.$$

Hence show that

$$J_{2m} = \frac{2m-1}{2m}.\frac{2m-3}{2m-2}.\cdots.\frac{1}{2}.\frac{\pi}{2};$$

$$J_{2m+1} = \frac{2m}{2m+1}.\frac{2m-2}{2m-1}.\cdots.\frac{2}{3}.$$

Show that $J_{2m+1} \le J_{2m} \le J_{2m-1}$, and hence that

$$\frac{2^2}{3^2}.\frac{4^2}{5^2}.\cdots.\frac{(2m-2)^2}{(2m-1)^2} \to \frac{\pi}{2} \qquad (m \to \infty).$$

Deduce that

$$\binom{2m}{m}.\frac{1}{2^{2m}} \sim \frac{1}{m\pi} \qquad (m \to \infty).$$

Q2. By the method of III.1, or otherwise, show that if $I_n := \int_0^{2\pi} \cos^{2n}\theta \, d\theta$, then

$$I_n = \binom{2n}{n}.\frac{1}{2^{2n}}.2\pi.$$

Obtain also the reduction formula

$$I_n = \frac{2n-1}{2n}.I_{n-1}$$

(from which Wallis' product may also be obtained, as in Q1).

Q3. Show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{1+n+n^2} = \frac{2\pi}{\sqrt{3}} \tanh(\pi\sqrt{3}/2).$$

Q4. Evaluate

$$\int_0^\infty \frac{\sin mx}{x} dx$$

for all real m.

NHB