## M2PM3 SOLUTIONS 1. 22.1.2010

Q1. (i) If x is a terminating decimal, x is a rational of the form  $n + m/10^k$ . If x is a recurring decimal, say

 $x = n.a_1 \dots a_k b_1 \dots b_\ell \dots b_1 \dots b_\ell \dots,$ 

x is  $n.a_1...a_k$  (rational, above) +y, where writing

$$b := b_1/10 + \dots b_\ell/10^\ell$$

(rational, above), y is a geometric series with first term  $b/10^k$  and common ratio  $10^{-\ell}.$  So

$$y = b.10^{-k} / (1 - 10^{-\ell}),$$

rational, so x is rational.

If x is rational, x = m/n say:

(a) take off its integer part – so reducing to  $0 \le m < n$ ,

(b) cancel m/n down to its lowest terms.

(ii) Now find the decimal expansion of m/n by the Long Division Algorithm. Let the remainders obtained by  $r_1, r_2, \ldots$  The expansion *terminates* if some  $r_k = 0$ . It *recurs* if some remainder has *already occurred*. As there are only n-1 different possible non-zero remainders, the expansion must terminate (with remainder 0) or recur (with a remainder the first repeat of one of  $1, 2, \ldots, n-1$ ) after at most n-1 places.

(The examples 1/7, 2/7, 3/7, 4/7, 5/7, 6/7 show that all n-1 places may be needed.)

(iii) Similarly with 10 replaced by 2,3, ...

Q2. 
$$f(x) = \exp(-1/x^2)$$
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(i)  
 $f' = -\frac{2}{x^3}e^{-1/x^2}, \qquad f'' = \frac{6}{x^5}e^{-1/x^2} - \frac{2}{x^3}\cdot\left(-\frac{2}{x^3}\right)e^{-1/x^2}.$ 

If inductively  $f^{(n)}(x) = P_n(1/x)e^{-1/x^2}$  with  $P_n$  a polynomial (of degree 3n),

$$f^{(n+1)}(x) = -\frac{1}{x^2} P'_n(1/x) e^{-1/x^2} - \frac{2}{x^3} P_n(1/x) e^{-1/x^2} = P_{n+1}(1/x) e^{-1/x^2},$$

completing the induction with  $P_{n+1}(x) = -x^2 P'_n(x) - 2x^3 P_n(x)$  (by induction,  $P_n$  has degree 3n).

(ii) f(0) = 0.

$$\frac{f^{(n+1)}(x) - f^{(n+1)}(0)}{x - 0} = \frac{1}{x} P_{(n+1)}(1/x) e^{-1/x^2} \to 0 \qquad (x \to 0),$$

as  $x^{-1}P_{(n+1)}(1/x) = yP_{(n+1)}(y)$  is a polynomial in  $y := 1/x (\to \infty \text{ as } x \to 0)$ ,  $e^{-1/x^2} = e^{-y^2}$ , but exponential growth is faster than polynomial growth. (iii) The Taylor series of f about 0 at x is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} 0.x^n / n! = \sum_{n=0}^{\infty} 0 = 0,$$

for all x. This = f(x) only at x = 0. Note. In Ch. II, §7, The Cauchy-Taylor Theorem, we shall see how to avoid such bad behaviour.

Q3.  $(\theta, \phi)$  represents a point uniquely, *except* at the North Pole  $\theta = 0$ , when any value of  $\phi$  will do. For, near the North Pole, we can approximate spherical polars on the sphere by plane polars on the tangent plane to the sphere at the North Pole. Plane polars  $(r, \theta)$  are not unique (only) at the origin: if r = 0, any  $\theta$  will do.

Another way of putting this is that pointing South is facing towards the South Pole. This is a unique direction at any point except the poles: at the North Pole, any direction faces South.

Q4. (i) For x = 0,  $f_n(x) = 0 \to 0$ . For x > 0,  $f_n(x) \sim 1/(nx) \to 0$ . (ii)  $(1 + n^2 n^2) = nn 2n^2 n - n(1 - n^2 n^2)$ 

$$f'_n(x) = \frac{(1+n^2x^2)n - nx \cdot 2n^2x}{(1+n^2x^2)^2} = \frac{n(1-n^2x^2)}{(1+n^2x^2)^2} = 0$$

where x = 1/n. One can check this is a maximum. Then

$$\sup_{[0,\infty)} f_n(.) = f_n(1/n) = \frac{1}{1+1} = \frac{1}{2},$$

which does not tend to 0 as  $n \to \infty$ . (iii)  $f_n \to 0$  uniformly iff  $\sup_{[0,\infty)} f_n \to 0$ .

Q5. Draw the unit circle C, and a regular hexagon with vertices  $P_0, P_1, \ldots, P_5$ with polar coordinates  $(r, \theta)$ , r = 1,  $\theta = 0, \pi/3, \ldots, 5\pi/3$ . With O the origin, triangle  $P_1OP_2$  is equilateral: angle  $P_1OP_2$  is  $\pi/3$  as this is a sixth of  $2\pi$  and the hexagon is regular; the other two angles are equal by symmetry, so both are also  $\pi/3$  as they sum to  $2\pi$ . So all sides of the triangle are equal – to 1, as  $P_1$ ,  $P_2$  are on the unit circle. Let  $P_1P_2$  cut the y-axis in Q. By symmetry,  $P_1Q = P_2Q = 1/2$ . Let the perpendicular from  $P_1$  to the x-axis meet Ox in R. Then  $ORP_1Q$  is a rectangle (all angles right angles), so  $OR = P_1Q = 1/2$ . The right-angled triangle  $ORP_1$  has angle  $\pi/3$  at O, so  $\cos(\pi/3) = OR/OP_1 = 1/2$ , and similarly (or by Pythagoras' Theorem)  $\sin(\pi/3) = \sqrt{3}/2$ .

NHB