M2PM3 SOLUTIONS 2. 28.1.2010

Q1. (i) At the origin $r = 0$, the argument θ is not defined uniquely – it can be anywhere in $[0, 2\pi]$.
(ii) In spherical polars (r, θ, ϕ), r is distance from the origin, and the angles are latitude and longitude (actually, colatitude and longitude). At the North Pole, longitude is not uniquely defined – any way you look, you are facing South.
Note. 1. This is connected with the special role of the North Pole in stereographic projections.
2. Near the North Pole, the Earth's surface is approximately flat, and one can use plane polar coordinates as local coordinates. Then non-uniqueness in (ii) reduces to non-uniqueness in (i).

Q2. (i) Put $x = ay$:

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + a^2} = \int_{-\infty}^{\infty} \frac{dy}{a^2(1 + y^2)} = \frac{1}{a}[\tan^{-1} y]_{\infty}^{\infty} = \frac{1}{a}(\pi/2 - (-\pi/2)) = \pi/a.$$

(ii)

$$\frac{1}{(x^2 + a^2)(x^2 + b^2)} = \frac{1}{(b^2 - a^2)}\Big(\frac{1}{(x^2 + a^2)} - \frac{1}{(x^2 + b^2)}\Big) \qquad (a \neq b).$$

This integrates to

$$\frac{1}{(b^2 - a^2)}\Big(\frac{\pi}{a} - \frac{\pi}{b}\Big) = \frac{\pi}{ab}\frac{(b - a)}{(b^2 - a^2)} = \frac{\pi}{ab(a + b)}.$$

If $a = b$: let $b \to a$ in the above. The integral $\to \pi/(a^2.2a) = \pi/(2a^3)$. So the answer holds for $a = b$ also. (We shall return to this example later as an application of Cauchy's Residue Theorem. We note its real-variable proof now.)

Q3. (i)

$$\begin{aligned} F(t) &= \int_0^{\infty} e^{-x} \cos xt dx = -\int_0^{\infty} \cos xt de^{x} \\ &= -[\cos xt.e^{-x}]_0^{\infty} + \int_0^{\infty} e^{-x}(-t \sin xt) dx \\ &= 1 = t \int_0^{\infty} \sin xt de^{-x} \\ &= 1 + t[\sin xt.e^{-x}]_0^{\infty} - t \int_0^{\infty} e^{-x}.t \cos xt dx \\ &= 1 - t^2 \int_0^{\infty} e^{-x} \cos xt dx = 1 - t^2 F(t) : \end{aligned}$$

$$F(t)(1 + t^2) = 1, \qquad F(t) = 1/(1 + t^2).$$

(ii)

$$\int_{-\infty}^{\infty} e^{ixt}.\frac{1}{2}e^{-|x|}dx = \int_{-\infty}^{\infty} \cos xt.\frac{1}{2}e^{-|x|}dx + i\int_{-\infty}^{\infty} \sin xt.\frac{1}{2}e^{-|x|}dx$$

$$= \int_{-\infty}^{\infty} \cos xt.\frac{1}{2}e^{-|x|}dx = 1/(1+t^2),$$

by (i) (the second integral is zero: *odd* integrand, symmetric limits. The first integral is twice $\int_0^\infty$: *even* integrand, symmetric limits.

Note. 1. Again, we will return to this later in a complex setting, but note this real-variable proof now.

2. In probabilistic language, this finds the characteristic function of the *symmetric exponential* probability density $\frac{1}{2}e^{-|x|}$ as $1/(1+t^2)$.

NHB

For Questions 1-4, a useful reference is

W. RUDIN, *Principles of mathematical analysis*, 3rd ed., McGraw-Hill, 1976.

Q1. If f is continuous on X under the ϵ, δ definition of being continuous at every point of X:

Choose any open set V in Y. We have to show $f^{-1}(V)$ open in X, i.e. that every point of $f^{-1}(V)$ is an interior point of $f^{-1}(V)$. If $f^{-1}(V)$ is empty, it is open; if not, it contains some point p, and then $f(p) \in V$. As V is open, there exists $\epsilon > 0$ such that $y \in V$ if $d(f(p), y) < \epsilon$. Since f is continuous at p, there exists $\delta > 0$ such that $d(f(x), f(p)) < \epsilon$ if $d(x,p) < \delta$. So if $d(x,p) < \delta$, then $d(f(x), f(p)) < \epsilon$, so $f(x) \in V$, i.e. $x \in f^{-1}(V)$: x is an interior point of $f^{-1}(V)$, as required.

Conversely, if $f^{-1}(V)$ is open for every open set $V \subset Y$:
Choose $p \in X$, $\epsilon > 0$; write $V := \{y : d(y, f(p)) < \epsilon\}$. This is open (in Y), so by assumption $f^{-1}(V)$ is open (in X). So there exists $\delta > 0$ such that for all x with $d(p,x) < \delta$, $x \in f^{-1}(V)$. But then $d(f(x), f(y)) < \epsilon$, so f is continuous under the ϵ, δ definition. (See Rudin, Th. 4.8, p.86-7.)

Q2. (i) The complement of an inverse image is the inverse image of the complement (check).

(ii) A set is open iff its complement is closed (lectures).

The function f is continuous iff inverse images of open sets are open (Q1)
iff complements of inverse images of open sets are closed (by (ii))
iff inverse images of complements of open sets are closed (by (i))
iff inverse images of closed sets are closed (by (i)),
as required. (See Rudin, Corollary, p.87.)

Q3. Let $\{G_\alpha\}$ be any open covering of $f(A)$. For each $a \in A$, $f(a)$ belongs to some G_α, so a belongs to some $f^{-1}(G_\alpha)$. That is, $\{f^{-1}(G_\alpha)\}$ is an open coveing of A, and A is compact. So there is some finite subcovering of A, $f^{-1}(G_1), \ldots, f^{-1}(G_n)\}$, say:

$$A \subset f^{-1}(G_1) \cup \ldots \cup f^{-1}(G_n).$$

That is,

$$f(A) \subset G_1 \cup \ldots \cup G_n.$$

So each open covering of $f(A)$ has a finite subcovering. So $f(A)$ is compact (Rudin, Th. 4.14, p.89).

Q4. $[a, b] \subset \mathbf{R}$ is closed and bounded, so (Heine-Borel Theorem) compact. As f is continuous, $f([a, b])$ is compact by Q3, so (closed and) bounded (Rudin, Th. 4.15, p. 89).