M2PM3 SOLUTIONS 4. 18.2.2010

Q1. *Triangle Lemma.*

Draw the line joining z_1 and z_2, and produce it until it meets triangle Δ – at points Z_1, Z_2 say. Then

$$|z_1 - z_2| \leq |Z_1 - Z_2|,$$

with equality iff both z_1, z_2 are *on* Δ rather than inside it (so $z_1 = Z_1$, $z_2 = Z_2$). There are two cases.

(i) Z_1, Z_2 lie on different sides of the triangle. Let Z_3 be the vertex in which these sides meet. Then by the Triangle Inequality,

$$|Z_1 - Z_2| \leq |Z_1 - Z_3| + |Z_2 - Z_3| \leq L_1 + L_2 \leq L,$$

where L_1, L_2 are the lengths of the sides containing Z_1, Z_2. Combining, $|z_1 - z_2| \leq L$.

(ii) Z_1, Z_2 lie on the same side, of length L_{12} say. Then

$$|Z_1 - Z_2| \leq L_{12} \leq L,$$

and the result follows as in (i).

Q2. *Harmonic conjugates.*

(i) For $u = x^3 - 3xy^2 - 2y$: $u_x = 3x^2 - 3y^2$, $u_{xx} = 6x$; $u_y = -6xy - 2$, $u_{yy} = -6x$; $u_{xx} + u_{yy} = 6x - 6x = 0$. So u is harmonic.

$v_y = u_x = 3x^2 - 3y^2$. Integrate wrt y: $v = 3x^2y - y^3 + F(x)$. Differentiate wrt x: $v_x = -u_y = 6xy + 2 = 6xy + F'(x)$. So $F'(x) = 2$, $F(x) = 2x + c$ (w.l.og. take $c = 0$). So $v = 3x^2y - y^3 + 2$; $f = u + iv = x^3 - 3xy^2 - 2y + 3ix^2y - iy^3 + 2ix = (x + iy)^3 + 2i(x + iy)$: $f(z) = z^3 + 2iz$.

(ii) For $u = x - xy$, $u_{xx} = 0$, $u_{yy} = 0$, so u is harmonic.

$v_y = u_x = 1 - y$. Integrate wrt y: $v = y - y^2/2 + F(x)$. Differentiate wrt x: $v_x = F'(x) = -u_y = x$, $F(x) = x^2/2$, $v = y - y^2/2 + x^2/2$; $f = u + iv = x - xy + iy + ix^2/2 - iy^2/2 = (x + iy) + \frac{1}{2}i(x + iy)^2$: $f = z + iz^2/2$.

Q3. *Union of Domains.*

Suppose $\bigcup_i D_i = G \cup H$ with G, H disjoint and open. We have to show one of G, H is empty. Now

$$D_j = (D_j \cap G) \cup (D_j \cap H).$$

The union on the RHS is disjoint (as G, H are), and the sets on RHS are open. As D_j is connected, one of these sets must be empty: say, $D_j \cap H$ is empty, i.e. $D_j \subset G$. Similarly, each D_k is contained in one of G, H. But if $D_k \subset H$, $D_j \cap D_k$ non-empty contradicts $G \cap H$ empty. So all the $D_k \subset G$. So H is empty.

Alternative Proof [assuming equivalence of connectedness and polygonal, or arcwise, connectedness].

Take $z_0 \in \bigcap_i D_i$. We can join any point in any D_j to z_0 by a path [e.g., a polygonal arc] lying in D_j, so in $\bigcup_i D_i$. So we can join any two points in $\bigcup_i D_i$ by such a path, by joining the paths linking each to z_0. So $\bigcup_i D_i$ is polygonally connected, so connected.

Q4 *Connected Components.*
Let z be any point in S. Let $\{C_i\}$ be the class of all connected subsets of S containing z. This class is non-empty (as $\{z\}$ is connected). By Q3, $C := \bigcup_i C_i$ is a connected subset of S containing z. By construction (as the union of all ...), C is maximal. So C is a component, and contains z. If C' is another component containing z, C' must be one of the C_i, so $C' \subset C$. But C' is maximal (it is a component). So $C \subset C'$, so $C = C'$. So each point z lies in a unique (connected) component, called the *(connected) component containing* z.

Q5. Write a_n for the coefficient of z^n.
(i) $a_{n+1}/a_n = -n/(n+1) = -1/(1+1/n) \to -1$, so $|a_{n+1}/a_n| \to 1$. So by the Ratio Test, the radius of convergence is 1. So the function is holomorphic in the unit disc $D := \{z : |z| < 1\}$.
Note. The sum function is $\log(1+z)$. This has a singularity at $z = -1$ (a branch point).
(ii) $a_{5n} = 1$, $a_{5n+k} = 0$ $(k = 1, 2, 3, 4)$. $\limsup |a_n|^{1/n} = 1$. So the radius of convergence is 1, and the region of holomorphy is again D.
Note. The sum function is $1/(1-z^5)$. This has 5 singularities on the unit circle, at the 5 fifth roots of unity.
(iii) $a_n = 1/n^n$, $a_n^{1/n} = 1/n \to 0$. So the radius of convergence is infinite. The sum function is holomorphic throughout the complex plane $\mathbf{C}$ (is an *entire* function, or an *integral* function).

NHB