M2PM3 SOLUTIONS 5. 25.2.2010

Q1 (i). $u = x^2 - y^2 - x$. So
$u_x = 2x - 1 = v_y$: $v = 2xy - y + F(x)$. So $v_x = 2y + F'(x) = -u_y = 2y$: $F'(x) = 0$, $F(x) = c$, constant, which we can take w.l.o.g. to be 0. So

$$v = 2xy - y,$$

$f = u + iv = x^2 - y^2 - x + 2ixy - iy = (x + iy)^2 - (x + iy) = z^2 - z$:

$$f(z) = z^2 - z.$$

(ii) The x term in u clearly comes from $z = x + iy$, so it suffices to deal with the $y/(x^2 + y^2)$ term.

So assume for now that $u = y/(x^2 + y^2)$. Then

$$u_x = -\frac{2xy}{(x^2 + y^2)^2} = v_y, \qquad v = -x \int \frac{2y\ dy}{(x^2 + y^2)^2}.$$

We can integrate this by the substitution $t = y^2$, giving

$$v = \frac{x}{t + x^2} + F(x) = \frac{x}{x^2 + y^2} + F(x).$$

So

$$v_x = \frac{1}{x^2 + y^2} - \frac{2x^2}{(x^2 + y^2)^2} + F'(x) = \frac{y^2 - x^2}{(x^2 + y^2)^2} + F'(x).$$

But

$$u_y = \frac{1}{x^2 + y^2} - \frac{2y^2}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2} = -v_x :$$

$$v_x = \frac{y^2 - x^2}{(x^2 + y^2)^2}.$$

Comparing, $F(x) = 0$, so F is constant, w.l.o.g. 0, so

$$v = \frac{x}{x^2 + y^2}.$$

So the $f = u + iv$ here is

$$\frac{y}{x^2 + y^2} + i\frac{x}{x^2 + y^2} = \frac{y + ix}{x^2 + y^2} = i\frac{x - iy}{x^2 + y^2} = i\frac{\bar{z}}{z\bar{z}} = i/z.$$

Thus the original $f = u + iv$ is

$$f(z) = z - i/z.$$

Q2. Let $f(\theta) := \sin\theta/\theta$. By L'Hospital's Rule, $f(0) = 1$.

$$f'(\theta) = \frac{\theta\cos\theta - \sin\theta}{\theta^2}.$$

The denominator is positive. So it suffices to show that the numerator, $g(\theta)$ say, is negative on $(0, \pi/2)$. But

$$g'(\theta) = \cos\theta - \theta\sin\theta - \cos\theta = -\theta\sin\theta < 0 \qquad (0 < \theta < \pi),$$

as required.

Q3. Since $z^n - z_0^n = (z-z_0)(z^{n-1}+z^{n-2}z_0+\ldots+zz_0^{n-2}+z_0^{n-1})$, $(z^n-z_0^n)/(z-z_0)$ is holomorphic at z_0, hence so is f, and similarly for z_0^{-1}. Near 0, the z^{-n} term dominates. It suffices to look at $1/z^n(z-z_0)(z-z_0^{-1})$, which near 0 is

$$z^{-n}(z-z_0)^{-1}(z-z_0^{-1})^{-1} = z^{-n}(1-\frac{z}{z_0})^{-1}(1-zz_0)^{-1},$$

or

$$z^{-n}(1+\frac{z}{z_0}+\frac{z^2}{z_0^2}+\ldots)(1+zz_0+z^2z_0^2+\ldots).$$

The coefficient of z^{-1} on RHS is

$$\frac{1}{z_0^{n-1}}+\frac{1}{z_0^{n-3}}+\ldots+z_0^{n-3}+z_0^{n-1} = z_0^{-(n-1)}(1+z_0^2+\ldots+z_0^{2(n-1)}) = \frac{(1-z_0^{2n})}{(1-z_0^2)},$$

on summing the geometric series. This is

$$\frac{1}{e^{i(n-1)\alpha}}.\frac{e^{2ni\alpha}-1}{e^{2i\alpha}-1} = \frac{e^{ni\alpha}-e^{-ni\alpha}}{e^{i\alpha}-e^{-i\alpha}} = \frac{\sin n\alpha}{\sin\alpha}.$$

Q4. Assume $\Gamma(-n+\zeta) \sim (-)^n/n!\zeta$. Then using $\Gamma(z+1) = z\Gamma(z)$ with $z = -n-1-\zeta$,

$$\Gamma(-n-1-\zeta) = \Gamma(-n+\zeta)/(-n-1-\zeta) \sim \frac{1}{(-n-1)}.\frac{(-)^n}{n!\zeta} = \frac{(-)^{n+1}}{(n+1)!\zeta},$$

completing the induction. For the second statement, $\Gamma(n+1-\zeta) \to \Gamma(n+1) = n!$ for $n = 0, 1, 2, \ldots$, so this follows from the first part for $n = 0, 1, 2, \ldots$. For negative n, interchange n and $-n$, z and $1-z$. For the last part (included because of (ii) below), just replace n by $-n$.
(ii) If $z = \pi(n+\zeta)$, $\sin\pi z = \sin\pi(n+\zeta) = \sin n\pi\cos\pi\zeta + \cos n\pi\sin\pi\zeta = (-)^n\sin\pi\zeta \sim (-)^n\pi\zeta$ as $\zeta \to 0$.

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