M2PM3 SOLUTIONS 6. 11.3.2010

Q1. As f is holomorphic for all $|z| \geq 1$ (including $+\infty$), $f(1/z)$ is holomorphic, and so continuous, in the closed unit disc $\bar{D} := \{z : |z| \leq 1\}$. So as $\bar{D}$ is compact, $f(1/z)$ is bounded on $\bar{D}$: $|f(1/z)| \leq M_1$, say, for $|z| \leq 1$, or $|f(z)| \leq M_1$ for $|z| \geq 1$. Similarly, as $f(z)$ is holomorphic, so continuous, in $\bar{D}$, f is bounded on $\bar{D}$: $|f(z)| \leq M_2$, say, for $|z| \leq 1$. So if $M := \max(M_1, M_2)$, $|f(z)| \leq M$ for all z in $\mathbf{C}$: f is bounded. As f is also holomorphic in $\mathbf{C}$, so entire, f is constant, by Liouville's theorem.

So if f is entire and non-constant, f has a singularity at ∞.

Examples:

polynomials (non-constant – of degree ≥ 1);

exponentials (e^z, e^{z^2}, etc.);

trig functions ($\sin z$, $\cos z$), etc.

Q2. With γ the ellipse $x^2/a^2 + y^2/b^2 = 1$ parametrized by $x = a\cos\theta$, $y = b\sin\theta$, $\int_\gamma dz/z = 2\pi i$, by CIF with $f \equiv 1$, $a = 0$ (or by Cauchy's Residue Theorem when we meet it, since $1/z$ has residue 1 at 0). So as $z = a\cos\theta + ib\sin\theta$ gives $dz = (-a\sin\theta + ib\cos\theta)d\theta$,

$$2\pi i = \int_0^{2\pi} \frac{-a\sin\theta + ib\cos\theta}{a\cos\theta + ib\sin\theta} d\theta$$

$$= \int_0^{2\pi} \frac{(-a\sin\theta + ib\cos\theta)(a\cos\theta - ib\sin\theta)}{a^2\cos^2\theta + b^2\sin^2\theta} d\theta$$

$$= \int_0^{2\pi} \frac{(b^2 - a^2)\sin\theta\cos\theta + iab(\cos^2\theta + \sin^2\theta)}{a^2\cos^2\theta + b^2\sin^2\theta} d\theta.$$

Equating imaginary parts,

$$\int_0^{2\pi} \frac{ab}{a^2\cos^2\theta + b^2\sin^2\theta} d\theta = 2\pi,$$

whence the result on dividing by ab.

Q3. For $0 < x < 1$, the integrals for both $\Gamma(x)$ and $\Gamma(1-x)$ converge, and

$$\Gamma(x)\Gamma(1-x) = \int_0^\infty t^{x-1}e^{-t}dt. \int_0^\infty u^{-x}e^{-u}du.$$

Substituting $u = tv$, the second integral on the right is $t^{1-x}\int_0^\infty v^{-x}e^{-tv}dv$. Cancelling powers of t and changing the order of integration, the RHS becomes

$$\int_0^\infty v^{-x}dv. \int_0^\infty e^{-(1+v)t}dt \quad = \int_0^\infty v^{-x}.\frac{1}{1+v}dv. \int_0^\infty e^{-w}dw \quad (w := (1+v)t).$$

The w-integral is 1. Interchanging x and $1-x$ (which preserves the LHS, and so the RHS also) gives

$$\Gamma(x)\Gamma(1-x) = \int_0^\infty \frac{v^{x-1}}{1+v} dv.$$

Q4. Since $(\sin x)/x$ is bounded on $[0,\infty)$, there is no problem about the integral existing, and one can show that F is continuous on $[0,\infty)$; it is clearly decreasing. So $I = F(0) = F(0+)$, by continuity. Differentiating under the integral sign (which we may – given) gives

$$\begin{aligned}
F'(t) &= -\int_0^\infty e^{-xt}\sin x dx \\
&= \int_0^\infty e^{-xt} d\cos x \\
&= [e^{-xt}\cos x]_0^\infty - \int_0^\infty \cos x.(-t)e^{-xt}dx \\
&= -1 + t\int_0^\infty e^{-xt} d\sin x \\
&= -1 + t[e^{-xt}\sin x]_0^\infty - t\int_0^\infty \sin x.(-t)e^{-xt}dx \\
&= -1 + t^2\int_0^\infty e^{-xt}\sin x dx \\
&= -1 - t^2 F'(t) :
\end{aligned}$$

$$(1+t^2)F'(t) = -1, \qquad F'(t) = -1/(1+t^2).$$

Integrating, $F(t) = -\tan^{-1} t + C$. But $F(t) \to 0$ as $t \to \infty$: $C = +\tan^{-1}\infty = \pi/2$.

Q5.

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_{\gamma(a,R)} \frac{f(z)}{(z-a)^{n+1}} dz.$$

If $|f(z)| \leq M|z|^k$ for large $|z|$, since on $\gamma(a,R)$, $|z| \leq |a| + R$,

$$|f^{(n)}(a)| \leq \frac{n!}{2\pi}.\frac{M(|a|+R)^k.2\pi R}{R^{n+1}} = O(1/R^{n-k}) \to 0 \quad (R\to\infty) \quad \text{if } n > k.$$

So $f^{(n)}(a) = 0$ for all a, i.e. $f^{(n)} \equiv 0$, i.e. f is a polynomial of degree $\leq k$.

NHB