**M2PM3 SOLUTIONS 7. 18.3.2010**

Q1. With $z := e^{i\theta}$ and $\gamma$ the unit circle,

$$I = \int_\gamma \frac{\left(\frac{z-z^{-1}}{2i}\right)^2 \frac{dz}{iz}}{a + \frac{b}{2}\left(z + z^{-1}\right)} = \frac{i}{2b}\int_\gamma \frac{(z^2-1)^2\ dz}{z^2(z^2+(2a/b)z+1} = \frac{i}{2b}\int_\gamma F(z)\ dz,$$

say. The roots of the denominator, $\alpha$ and $\beta$ say, have product 1 (so $\beta = 1/\alpha$), and are given by

$$-\frac{a}{b} \pm \frac{1}{b}\sqrt{a^2-b^2}.$$

The $-$ gives the root of larger modulus, which is outside $\gamma$; we want the root of smaller modulus, inside $\gamma$, given by $+$: $\alpha$ say. Then

$$F(z) = \frac{(z^2-1)^2}{z^2(z-\alpha)(z-\beta)}.$$

By the Cover-Up Rule (or direct expansion),

$$Res_\alpha F = \frac{(a^2-1)^2}{\alpha^2(\alpha-\beta)} = \frac{(\alpha-1/\alpha)^2}{(\alpha-\beta)} = \frac{(\alpha-\beta)^2}{\alpha-\beta} = \alpha-\beta = 2\sqrt{a^2-b^2}/b.$$

Expanding $F(z)$ about $z = 0$ gives

$$F(z) = z^{-2}(1+(2a/b)z+z^2)^{-1}(1-z^2)^2 = z^{-2}(1-(2a/b)z+O(z^2)).$$

So picking out the coefficient of $z^{-1}$ (the residue), $Res_0 F = -2a/b$. So by CRT,

$$I = \frac{i}{2b}.2\pi i.(-\frac{2a}{b} + \frac{2\sqrt{a^2-b^2}}{b}): \qquad I = \frac{2\pi}{b^2}(a - \sqrt{a^2-b^2}).$$

Q2. As $x^4+5x^2+6 = (x^2+3)(x^2+2)$, which has simple zeros at $\pm i\sqrt{2}$, $\pm i\sqrt{3}$, use $f(z) := z^2/(z^2+3)(z^2+2)$ and the contour $\Gamma$ consisting of a large semicircle in the upper half-plane with base $[-R, R]$. Then $f$ has simple poles inside $\Gamma$ at $i\sqrt{3}$, $i\sqrt{2}$. By Jordan's Lemma, the integral round the semicircle tends to 0 as $R \to \infty$, while the integral along the base tends to the integral $I$ we are to evaluate. So by Cauchy's Residue Theorem, $I = \sum Res f$, the sum being over the poles at $i\sqrt{3}$ and $i\sqrt{2}$ inside $\Gamma$. As both poles are simple, we can use the Cover-Up Rule:

$$Res_{i\sqrt{3}} f = (i\sqrt{3})^2/[(-3+2)(i\sqrt{3}+i\sqrt{3})] = (-3)/[-2i\sqrt{3}] = -i\sqrt{3}/2;$$

$$Res_{i\sqrt{2}} f = (i\sqrt{2})^2/[(-2+3)(i\sqrt{2}+i\sqrt{2})] = (-2)/[2i\sqrt{2}] = i\sqrt{2}/2.$$

So by CRT,

$$I = 2\pi i \sum Res f = 2\pi i.(-)i(\sqrt{3}-\sqrt{2})/2 = \pi(\sqrt{3}-\sqrt{2}).$$

Q3. Use $f(z) = (e^{ipz} - e^{iqz})/z^2$. This has a pole at 0 (apparently double, but actually single: the numerator has a simple zero at 0). In the upper half-plane, we use a semicircular contour, with a small semi-circular indentation to avoid this pole – a contour $\Gamma$ consisting of:
(i) $\Gamma_1$, the line segment $[-R, -r]$ ($R$ large, $r > 0$ small);
(ii) $\Gamma_2$, the semi-circle centre 0 radius $r$, clockwise (-ve sense);
(iii) $\Gamma_3 = [r, R]$;
(iv) $\Gamma_4$, the semi-circle centre 0 radius $R$, anticlockwise (+ve sense).
On $\Gamma_4$, $|e^{ipz}| = |e^{ip(x+iy)}| = e^{-py} \leq 1$, as $p \geq 0$ and $y \geq 0$ in the upper half-plane, and similarly $|e^{iqz}| \leq 1$. So $|f(z)| = O(1/R^2)$, and by $(ML)$,
$\int_{\Gamma_4} f = O(1/R^2).\pi R = O(1/R) \to 0$ as $R \to \infty$.
As $R \to \infty, r \to 0$, $\int_{\Gamma_1} f + \int_{\Gamma_3} f \to I$, the required integral. On $\Gamma_2$, $z = re^{i\theta}$,

$$f(z) = [(1+ipz-p^2z^2/2...)-(1+iqz-q^2z^2/2...)]/z^2 = [i(p-q)-\frac{1}{2}(p^2-q^2)z+...]/z,$$

$dz/z = ire^{i\theta}d\theta/re^{i\theta} = id\theta$, where $\theta$ goes from $\pi$ to 0. So (changing the sign to interchange the limits of integration)

$$\int_{\Gamma_2} f = -\int_0^\pi [i(p-q) + O(r)](id\theta) \to \pi(p-q) \qquad (r \to 0).$$

Since $\int_\Gamma f = 0$ by Cauchy's Theorem, this gives $I + \pi(p-q) = 0$: $I = -\pi(p-q)$.

Q4. Use $f(z) = 1/(z^4 + a^4)$ and as contour $\gamma$ the interval $\gamma_1 := [-R, R]$, completed by a semi-circle $\gamma_2$ of radius $R$ in the upper half-plane. On $\gamma_2$, $|f| = O(1/R^4)$, so by ML $\int_{\gamma_2} f = O(1/R^3) \to 0$ as $R \to \infty$, while $\int_{\gamma_1} f \to 2I$ by symmetry. The integrand $f$ has poles where $z^4 = -a^4 = a^4e^{i\pi}$, $z = ae^{i\pi/4}$, $ae^{3i\pi/4}$, $ae^{5i\pi/4}$, $ae^{7i\pi/4}$; only the first two matter (are inside $\gamma$). If $\alpha$ is such a root $\alpha^4 = -a^4$, and we can evaluate the residue at $\alpha$ as

$$Res_\alpha f = \lim_{z\to\alpha} \frac{z-\alpha}{z^4-\alpha^4}.$$

As $z^4 - \alpha^4 = (z-\alpha)(z^3 + z^2\alpha + z\alpha^2 + \alpha^3)$, the RHS is

$$1/(z^3 + z^2\alpha + z\alpha^2 + \alpha^3) \to 1/(4\alpha^3) = \alpha/(4\alpha^4) = -\alpha/(4a^4) \qquad (z \to \alpha)$$

(by the Cover-Up Rule, or directly). So by CRT,

$$2I = 2\pi i.\frac{-1}{4a^4}(ae^{i\pi/4} + ae^{3i\pi/4}).$$

But

$$e^{i\pi/4} + e^{3i\pi/4} = e^{i\pi/4}(1 + e^{i\pi/2}) = \frac{1}{\sqrt{2}}(1+i).(1+i) = 2i/\sqrt{2} = i\sqrt{2}.$$

So $I = \sqrt{2}\pi/(4a^3)$. //

NHB