M2PM3 SOLUTIONS 8, 25.3.2010

Q1.

$$I_n = \int \sin^n x \, dx = -\int \sin^{n-1} x \, d\cos x = -\sin^{n-1} x \cos x + \int \cos x.(n-1)\sin^{n-2} x \cos x \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x (1-\cos^2 x) \, dx :$$

$$nI_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}.$$

Passing to the definite integral $J_n := \int_0^{\pi/2} \sin^n x \, dx$ gives

$$J_n = \frac{n-1}{n}.J_{n-2}.$$

So as $\int_0^{\pi/2} dx = \pi/2$, $\int_0^{\pi/2} \sin x \, dx = [-\cos x]_0^{\pi/2} = 1$,

$$J_{2m} = \frac{2m-1}{2m}.\frac{2m-3}{2m-2}.\ldots.\frac{1}{2}.\frac{\pi}{2}, \qquad J_{2m+1} = \frac{2m}{2m+1}.\frac{2m-2}{2m-1}.\ldots.\frac{2}{3}.$$

Dividing,

$$\frac{\pi}{2} = \frac{2}{1}.\frac{2}{3}.\frac{4}{3}.\frac{4}{5}.\ldots.\frac{2m}{2m-1}.\frac{2m}{2m+1}.\frac{J_{2m}}{J_{2m+1}}.$$

But as $0 \le \sin x \le 1$ in $[0, \pi/2]$, $\sin^{2m+1} x \le \sin^{2m} x \le \sin^{2m-1} x$; integrating gives $J_{2m=1} \le J_{2m} \le J_{2m-1}$. So

$$1 \le \frac{J_{2m}}{J_{2m-1}} \le \frac{J_{2m-1}}{J_{2m+1}} = 1 + \frac{1}{2m} \downarrow 1 : \qquad \frac{J_{2m}}{J_{2m+1}} \to 1 \qquad (m \to \infty).$$

So

$$\frac{\pi}{2} = \lim_{m\to\infty} \frac{2}{1}.\frac{2}{3}.\frac{4}{3}.\frac{4}{5}.\ldots.\frac{2m}{2m-1}.\frac{2m}{2m+1}.$$

As $2m/(2m+1) \to 1$, this gives

$$\frac{2^2}{3^2}.\frac{4^2}{5^2}.\ldots.\frac{(2m-2)^2}{(2m-1)^2} \to \frac{\pi}{2}.$$

Take square roots and multiply top and bottom by $2.4.\ldots.(2m-2).2m.\sqrt{2m}$. In the numerator $2^2.4^2.\ldots.(2m-2)^2(2m)^2 = 2^{2m}(m!)^2$, giving

$$\frac{2^{2m}(m!)^2}{(2m)!\sqrt{m}} \to \sqrt{\pi} : \qquad \binom{2m}{m}.\frac{1}{2^{2m}} \sim \frac{1}{\sqrt{m\pi}} \qquad (m \to \infty).$$

Note. See III.10 for a proof by Complex Analysis. This method is nearer to Wallis' in 1656 (see Dramatis Personae): Wallis evaluated $\int_0^1 (1-x^2)^n dx$ – pre-calculus (Newton' *Principia* appeared in 1687, and was influenced by Wallis).

Q2. Putting $z = e^{i\theta}$ as in III.1, $\cos\ \theta = (z + z^{-1})/2$, $d\theta = dz/(iz)$, so with γ the unit circle,

$$I_n = \int_0^{2\pi} \cos^{2n}\theta\ d\theta = \int_\gamma \frac{1}{2^{2n}}(z+1/z)^{2n}\frac{dz}{iz} = -\frac{i}{2\pi}\int_\gamma \sum_{k=0}^{2n}\binom{2n}{k} z^k.z^{k-2n}.dz/z$$

$$= -\frac{i}{2\pi}\int_\gamma \sum_{k=0}^{2n}\binom{2n}{k} z^{2k-2n}.dz/z.$$

By the Fundamental Integral (or directly now, by CRT), only the $k = n$ term on the RHS contributes; $\int_g dz/z = 2\pi i$, whence the result.

The reduction formula is proved by the same method as in Q1.

Q3. Put $f(z) = (\pi \cot \pi z)/(1 + z + z^2)$. Since $z^3 - 1 = (z-1)(z^2 + z + 1)$, the roots of $z^2 + z + 1$ are $e^{2\pi i/3} = -1/2 + i\sqrt{3}/2$ and $e^{4\pi i/3} = -1/2 - i\sqrt{3}/2$, the complex cube roots of unity other than 1. Integrating f round the square contour Γ_n with vertices $(n + 1/2)(\pm 1 \pm i)$ gives

$$\int_{\Gamma_n} f = 2\pi i\Big(\sum_{k=-n}^{n} \frac{1}{1+k+k^2} + Res_{e^{2\pi i/3}} f + Res_{e^{4\pi i/3}} f\Big).$$

$$\int_{\Gamma_n} f = O(1/n^2).O(n) = O(1/n) \to 0 \qquad (n \to \infty) \qquad \text{(by ML)}.$$

Combining,

$$\sum_{n=-\infty}^{\infty} \frac{1}{1+n+n^2} = -\Big(Res_{e^{2\pi i/3}} + Res_{e^{4\pi i/3}}\Big)\frac{\pi \cot \pi z}{(z - e^{2\pi i/3})(z - e^{4\pi i/3})}.$$

By the Cover-Up Rule, the RHS is

$$-\frac{\pi \cot(\pi e^{2\pi i/3})}{i\sqrt{3}} + \frac{\pi \cot(\pi e^{4\pi i/3})}{i\sqrt{3}} = \frac{i\pi}{\sqrt{3}}[(\cot(-\frac{\pi}{2} + \frac{i\pi\sqrt{3}}{2}) - \cot(-\frac{\pi}{2} - \frac{i\pi\sqrt{3}}{2})].$$

Since $\tan(a + \pi/2) = -\cot a$, $\cot(a - \pi/2) = -\tan a$, the RHS is

$$\frac{i\pi}{\sqrt{3}}[\tan(-\frac{i\pi\sqrt{3}}{2}) - \tan(\frac{i\pi\sqrt{3}}{2})] = \frac{2i\pi}{\sqrt{3}}\tan(-\frac{i\pi\sqrt{3}}{2}).$$

As $i \tan i\theta = \tanh\theta$, this is $(2i\pi/\sqrt{3}).(-i).\tanh(\pi\sqrt{3}/2)$. So

$$\sum_{n=-\infty}^{\infty} \frac{1}{1+n+n^2} = \frac{2\pi}{\sqrt{3}}\tanh(\pi\sqrt{3}/2).$$

Q4. For $m > 0$, put $u := mx$. Since $dx/x = du/u$, this reduces the problem to the case $m = 1$, which gives $I = \pi/2$ (Lectures). For $m < 0$, we get $I = -\pi/2$, since the integrand is odd in m. For $m = 0$, we get 0 since the integrand is 0.

NHB