m2pm3cw1soln(11).tex

# M2PM3 COMPLEX ANALYSIS: SOLUTIONS TO ASSESSED COURSEWORK 1, 2011

Th 10 Feb 2011

Q1. (i) As $\cos\ n\theta = T_n(\cos\theta)$, $|T_n(\cos\theta)| = |\cos\ n\theta| \le 1$. So as $-1 \le \cos\theta \le 1$, $|T_n(x)| \le 1$ for $x \in [-1,1]$: the bounds of $|T_n|$ on $[-1,1]$ are -1 and +1 **[1]**.
(ii) As $T_n$ is a polynomial, so continuous, $T_n$ is bounded (meaning $|T_n|$ is bounded) on compact sets in the $z$-plane, by Heine's Theorem. So $T_n$ is bounded on closed bounded sets in the $z$-plane, by the Heine-Borel Theorem. Since any bounded set has bounded closure, on which $T_n$ is bounded (as above), $T_n$ is bounded on any bounded set in the plane.

On the other hand, as $T_n$ is a polynomial of degree $n$ (with leading coefficient $2^{n-1}$), for $|z|$ large enough $T_n(z)$ is dominated by its leading term $(2^{n-1}z^n)$, which tends to $\infty$ in modulus as $|z| \to \infty$. Since we can let $z \to \infty$ through any unbounded set, $T_n$ is unbounded on any unbounded set.

Combining: the sets in the plane on which $T_n$ is bounded are exactly the bounded sets **[4]**.

Q2 **[5]**. For $k = 0$, $e^{2\pi ik/n} = 1$, so the order is 1. For the rest, $k = 1, 2, \ldots, n-1$, giving the complex $n$th roots of unity. Here the order of the $n$th roots of unity depends on the prime power factorisation of $n$. If $n$ is a prime $p$, all complex $n$th roots of unity have order $p$. If $n$ is composite, $k = 1$ gives a root of order $n$, and so does any $k$ coprime to $n$ (i.e., $k$ and $n$ have no common factors), but if $k$ is a divisor of $n$, $k = n/m$ say, then $k/n = 1/m$, giving a complex $m$th root of unity, with order $m$.
[In algebraic language, the $n$th roots of unity form a multiplicative group isomorphic to the additive group $\mathbf{Z}_n$ of integers modulo $n$. This is a cyclic group, and the above describes the subgroup structure of a cyclic group, as in Algebra or Group Theory.]

Q3. (i) The coefficients $a_n$ are 1 along the sequence of powers of 2, 0 otherwise. So $\limsup |a_n|^{1/n} = 1$, and the radius of convergence is 1 **[1]**.
(ii) $a_m = 0$ unless $m = 2^n$, when $a_m = 1/n!$ The radius of convergence is $R = 1/\limsup |a_m|^{1/m}$. So $1/R = \limsup (n!)^{1/m}$. By Stirling's formula,
$n! \sim \sqrt{2\pi}e^{-n}n^{n+1/2}$. As $m = 2^n$, $n = (\log m)/(\log 2)$ (so $\log m = \log\log m - \log\log 2$). So

$$(n^n)^{1/m} = (\exp(n\log n))^{1/m} = \exp(\frac{n\log n}{m})$$

$$= \exp(\frac{((\log m)/(\log 2))(\log\log m - \log\log 2)}{m}) \to 1 \quad (m \to \infty),$$

as $m \to \infty$ faster than $(\log m)(\log\log m)$. Similarly, $(e^{-n})^{1/m} \to 1$, etc. Combining, $(n!)^{1/m} \to 1$. So $R = 1$ **[2]**.

(iii) The same argument shows that here $R = 1$ also [**1**].

Q4. (i) (a) Since $1/(1+z) = 1 - z + z^2 - z^3 \ldots$,

$$\frac{z}{e^z - 1} = \frac{z}{z + z^2/2 + z^3/6 + \ldots} = 1/(1 + [z/2 + z^2/6 + \ldots])$$

$$= 1 - [...] + [...]^2 \ldots = 1 - z/2 - z^2/6 \ldots + z^2/4 \ldots = 1 - z/2 + z^2/12 \ldots :$$

$$\frac{z}{e^z - 1} = 1 - z/2 + z^2/12 \ldots \qquad [\mathbf{1}]$$

(The expansion coefficients here are essentially the *Bernoulli numbers*, important in Analytic Number Theory, Combinatorics and elsewhere.)

(b) The power series expansion converges where the denominator $e^z - 1$ is well-behaved (non-zero), i.e. except where $e^z = 1 = e^{2\pi in}$, $z = 2\pi in$ with $n$ integer. (The only kind of bad behaviour – singularity – possible for such a well-behaved function is a zero in the denominator, as here.) The singularity (simple zero in the denominator) is cancelled by the $z$ in the numerator at $z = 0$, i.e. for $n = 0$, but not elsewhere. So the nearest singularity to the origin is $z = \pm 2\pi i$ (where we shall see in Ch. II that the function has a simple pole of residue $\pm 2\pi i$). The distance from the origin to either singularity is $2\pi$, so:

the radius of convergence of the power series is $2\pi$ [**1**].

(We know that the power series converges inside a circle centre the origin. The radius $R$ of this circle can be at most $|z_0|$, the distance to the nearest singularity $z_0$ to the origin – if it were more, $z_0$ would be inside the circle of convergence, the power series would converge there, and it would not be a singularity. In fact, as assumed in the Sixth Form, and as we shall prove in Ch. II, $R = |z_0|$. You were not asked to prove this, so you may quote it here.)

(ii) (a)

$$1/\cos z = 1/(1 - z^2/2 + z^4/4! \ldots) = 1/(1 - [z^2/2 - z^4/4! \ldots]) = 1 + [...] + [...]^2 \ldots :$$

$$1/\cos z = 1 + z^2/2 + \ldots \quad \text{(the } z \text{ term is 0).} \qquad [\mathbf{1}]$$

(b) The singularities are where $\cos z = 0$, i.e. where $z = (n + \frac{1}{2})i\pi$. The nearest to the origin are $n = 0$ or $-1$, at $\pm i\pi/2$, distance $\pi/2$ from 0. So the radius of convergence is $\pi/2$ .

(iii) $\mathrm{ch}z := \frac{1}{2}(e^z + e^{-z})$, and $\cos z := \frac{1}{2}(e^{iz} + e^{-iz})$. So $\mathrm{ch}z = \cos(z/i) = \cos(-iz) = \cos iz$. So by (ii):

(a) the first three terms of the power-series expansion are $\mathrm{ch}z = 1 - z^2/2 \ldots$ (the $z$ term is 0) [**1**];

(b) the radius of convergence is $\pi/2$ [**1**].

NHB