m2pm3cw2(11).tex

M2PM3 COMPLEX ANALYSIS: ASSESSED COURSEWORK 2, 2011

Mon 21 Feb 2011, deadline 2pm Wed 2 March 2011

Q1 [**4**] (*Chain Rule*). If $g : D_1 \to D_2$ and $f : D_2 \to \mathbf{C}$ with f, g holomorphic and D_1, D_2 domains, show that$f(g) : D_1 \to \mathbf{C}$ is holomorphic, and its derivative is $(f(g))' = f'(g)g'$.

Q2 [**5**]. If $f : K \to \mathbf{C}$ with K compact is continuous, we know f is bounded on K, by Heine's Theorem. Show that f attains its bound on K: there exists $z_0 \in K$ with $|f(z_0)| = \sup\{|f(z)| : z \in K\}$.

Q3 [**4**]. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(z) = \sum_{n=0}^{\infty} b_n z^n$ have radii of convergence R_1, R_2, and $f(z)g(z) = \sum_{n=0}^{\infty} c_n z^n$ for $|z| < R := \min(R_1, R_2)$, find c_n.

Q4 [**2**]. In Q3, if $R > 1$, $A := \sum_{n=0}^{\infty} a_n$, $B := \sum_{n=0}^{\infty} b_n$, $C := \sum_{n=0}^{\infty} c_n$, show that $C = AB$.

Q5 [**5**]. By integrating by parts, or otherwise, show that if $I_n := \int \sin^n x \, dx$,

$$nI_n = -\sin^{n-1} x \cos\ x + (n-1)I_{n-2}.$$

Deduce that if $J_n := \int_0^{\pi/2} \sin^n x \, dx$, one has the *reduction formula*

$$J_n = \frac{n-1}{n}.J_{n-2}.$$

Hence show that

$$J_{2m} = \frac{2m-1}{2m}.\frac{2m-3}{2m-2}.\cdots.\frac{1}{2}.\frac{\pi}{2};$$

$$J_{2m+1} = \frac{2m}{2m+1}.\frac{2m-2}{2m-1}.\cdots.\frac{2}{3}.$$

Show that $J_{2m+1} \le J_{2m} \le J_{2m-1}$, and hence that

$$\frac{2^2}{3^2}.\frac{4^2}{5^2}.\cdots.\frac{(2m-2)^2}{(2m-1)^2}.2m \to \frac{\pi}{2} \qquad (m \to \infty).$$

Deduce that

$$\binom{2m}{m}.\frac{1}{2^{2m}} \sim \frac{1}{\sqrt{m\pi}} \qquad (m \to \infty).$$

NHB