M2PM3 COMPLEX ANALYSIS: SOLUTIONS TO ASSESSED COURSEWORK 2, 2011

Th 3 March 2011

Q1 (*Chain Rule*). For $h \to 0$, $g(z+h) = g(z) + hg'(z) + o(h)$. So

$$\begin{aligned} f(g(z+h)) - f(g(z)) &= f(g(z) + hg'(z) + o(h)) - f(g(z)) \\ &= f'(g(z)).(hg'(z) + o(h)) + o(hg'(z) + o(h)) \\ &= f'(g(z))g'(z)h + o(h). \end{aligned}$$

This says that $(f(g))'$ exists at z and is $f'(g(z)).g'(z)$, as required. **[4]**
Note. 1. One can write this out at greater length using ϵ, but the above is just as rigorous, shorter and clearer.
2. No need for any division (e.g., by $g(z+h) - g(z)$, or $g'(z)$ – these may be zero!).

Q2. If $M := \sup\{|f(z)| : z \in K\}$, there exist $z_n \in K$ with $|f(z_n)| \to M$. As K is compact, the sequence z_n has a convergent subsequence, $z_{n(k)}$ say. For:
(i) K is compact, so closed and bounded by the Heine-Borel Theorem, so any infinite set of points in K has a convergent subsequence, by the Bolzano-Weierstrass Theorem, or
(ii) In a metric space, compactness is the same as sequential compactness, which means that any infinite sequence contains a convergent subsequence.

Write z_0 for the limit of $z_{n(k)}$ as $k \to \infty$. Then $z_0 \in K$, as K is compact, so closed (by Heine-Borel – or, a compact subset of a Hausdorff space – in particular, of a metric space – is closed). As $|f(z_n)| \to M$, $|f(z_{n(k)})| \to M$. As f is continuous and $z_{n(k)} \to z_0$, $f(z_{n(k)}) \to f(z_0)$, so $|f(z_{n(k)})| \to |f(z_0)|$. Combining, $|f(z_0)| = M$ and $z_0 \in K$. **[5]**

Q3. For $|z| < R$, i.e. inside both circles of convergence,

$$f(z)g(z) = (\sum_{j=0}^{\infty} a_j z^j)(\sum_{k=0}^{\infty} c_k z^k) = \sum \sum a_j b_k z^{j+k}.$$

Both power series converge absolutely inside their circles of convergence, so by absolute convergence we may write $n := j + k$ ($n = 0, 1, \dots$) and re-arrange the RHS, to get

$$f(z)g(z) = \sum_{n=0}^{\infty} z^n \sum_{j+k=n} a_j c_k.$$

So

$$c_n = \sum_{j+k=n} b_j c_k = \sum_{j=0}^{n} a_j b_{n-j} = \sum_{k=0}^{n} a_{n-k} b_k. \qquad \textbf{[4]}$$

[The sequence $c = (c_n)$ is called the *convolution* of (a_n) and (b_n).]

Q4. If $R > 1$, we may put $z = 1$ in Q3 to get $f(1)g(1) = (\sum_0^\infty a_m)(\sum_0^\infty b_n) = AB$ on the left, and $C = \sum_0^\infty c_n$ on the right. So $AB = C$. **[2]**
Note. In fact $AB = C$ is still true if only *one* of A, B absolutely convergent. This is *Mertens' Theorem*, which we quote.]

Q5 [**5**].

$$I_n = \int \sin^n x \, dx = -\int \sin^{n-1} x \, d\cos x = -\sin^{n-1} x \cos x + \int \cos x.(n-1)\sin^{n-2} x \cos x \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \cos^2 x) \, dx :$$

$$nI_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}. \qquad [\mathbf{1}]$$

Passing to the definite integral $J_n := \int_0^{\pi/2} \sin^n x \, dx$ gives

$$J_n = \frac{n-1}{n}.J_{n-2}. \qquad [\mathbf{1}]$$

So as $\int_0^{\pi/2} dx = \pi/2$, $\int_0^{\pi/2} \sin x \, dx = [-\cos x]_0^{\pi/2} = 1$,

$$J_{2m} = \frac{2m-1}{2m}.\frac{2m-3}{2m-2}.\ldots.\frac{1}{2}.\frac{\pi}{2}, \qquad J_{2m+1} = \frac{2m}{2m+1}.\frac{2m-2}{2m-1}.\ldots.\frac{2}{3}. \qquad [\mathbf{1}]$$

Dividing,

$$\frac{\pi}{2} = \frac{2}{1}.\frac{2}{3}.\frac{4}{3}.\frac{4}{5}.\ldots.\frac{2m}{2m-1}.\frac{2m}{2m+1}.\frac{J_{2m}}{J_{2m+1}}.$$

But as $0 \le \sin x \le 1$ in $[0, \pi/2]$, $\sin^{2m+1} x \le \sin^{2m} x \le \sin^{2m-1} x$; integrating gives $J_{2m+1} \le J_{2m} \le J_{2m-1}$. So

$$1 \le \frac{J_{2m}}{J_{2m+1}} \le \frac{J_{2m-1}}{J_{2m+1}} = 1 + \frac{1}{2m} \downarrow 1 : \quad \frac{J_{2m}}{J_{2m+1}} \to 1 : \quad \frac{J_{2m+1}}{J_{2m}} \to 1 \quad (m \to \infty).$$

So

$$\frac{\pi}{2} = \lim_{m\to\infty} \frac{2}{1}.\frac{2}{3}.\frac{4}{3}.\frac{4}{5}.\ldots.\frac{2m}{2m-1}.\frac{2m}{2m+1} : \quad \frac{2^2.4^2.\ldots.(2m-2)^2}{3^2.5^2.\ldots.(2m-1)^2}.2m \to \frac{\pi}{2}.$$

[**1**]

Take square roots and multiply top and bottom by $2.4.\ldots.(2m-2) = 2^{m-1}(m-1)!$:

$$\frac{2.4.\ldots.(2m-2)}{3.5.\ldots.(2m-1)}.\sqrt{2m} \to \sqrt{\frac{\pi}{2}} : \quad \frac{2^{2(m-1)}(m-1)!^2}{(2m-1)!}\sqrt{2m} \to \sqrt{\frac{\pi}{2}}.$$

Multiplying top and bottom by $(2m)^2$,

$$\frac{2^{2m} m!^2}{(2m)!}.\frac{\sqrt{2m}}{2m} \to \sqrt{\frac{\pi}{2}} : \quad 2^{-2m}\binom{2m}{m}\sqrt{2m} \to \sqrt{\frac{2}{\pi}} : \quad 2^{-2m}\binom{2m}{m} \sim \frac{1}{\sqrt{m\pi}} \qquad (m \to \infty).$$

[**1**]

Note. 1. This is *Wallis' product* for π; see III.9 for a proof by Complex Analysis.
2. The last part is most easily checked by using *Stirling's formula.* NHB