M2PM3 EXAMINATION 2009

Q1. (i) If $z = 1 + i\sqrt{3}$, find (a) z in polar form; (b) z^5 in polar and cartesian forms; (c) z^{-1} in polar and cartesian forms.
(ii) Find and classify the singularities of $f(z) = 1/(1 + \cosh\ z)$.
(iii) If f has a pole of order k at a, show that f' has a pole of order $k + 1$ at a.

Q2. (i) Define a *star-domain* D with *star-centre* z_0.
(ii) Prove the theorem of the primitive, that if f is holomorphic in a star-domain D with star-centre z_0, then (with $[z_0, z]$ the line segment from z_0 to z)

$$F(z) := \int_{[z_0,z]} f(w)dw$$

has $F' = f$ in D (you may quote Cauchy's theorem for triangles).
(iii) Let

$$f(z) := \int_{[1,z]} dw/w \qquad (z \in D),$$

where D is the largest star-domain with star-centre 1 for which the above defines f as a convergent integral.
(a) Find D.
(b) Show that if $g(z) := e^z$, $h(z) := f(g(z))$, then

$$h'(z) = 1, \qquad h(z) = z.$$

Q3. (i) State and prove the Cauchy-Taylor theorem for a function f holomorphic in an open disc D centre a radius R (you may quote the Cauchy integral formula).
(ii) For a real, define $\binom{a}{n} := a(a-1)\dots(a-n+1)/n!$ Show that

$$(1+z)^a = \sum_{n=0}^{\infty} \binom{a}{n} z^n \qquad (|z| < 1).$$

Check that the radius of convergence of the power series on the right is indeed 1.

Q4. (i) For γ the circle centre 0 radius 2, find

$$\int_\gamma \frac{\sin z}{(z^2+1)}\ dz.$$

(ii) Show that for $p, q \geq 0$,

$$\int_{-\infty}^{\infty} \frac{\cos px - \cos qx}{x^2}\ dx = \pi(q-p).$$

Hence or otherwise show that

$$\int_0^\infty (\frac{\sin x}{x})^2\ dx = \frac{\pi}{2}.$$

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