M2PM3 EXAMINATION 2010

Q1. (i) [**9**] By de Moivre's theorem, or otherwise,
(a) express cos $n\theta$, sin $n\theta$ as polynomials in $c := \cos\theta$, $s := \sin\theta$; [**3,3**]
(b) express tan $n\theta$ as a rational function in $t := \tan\theta$. [**3**]
(ii) [**4**] Show that the roots of the polynomial equation

$$7 - \binom{7}{3}t^2 + \binom{7}{5}t^4 - t^6 = 0$$

are $\tan\pi/7$, $\tan 2\pi/7$, ..., $\tan 6\pi/7$.
(iii) [**7**] By considering $\int_\gamma dz/z$ with γ the ellipse $x^2/a^2 + y^2/b^2 = 1$ $(a,\, b > 0)$ parametrized by $x = a\cos\theta$, $y = b\sin\theta$, or otherwise, show that

$$\int_0^{2\pi} \frac{d\theta}{a^2\cos^2\theta + b^2\sin^2\theta} = \frac{2\pi}{ab}.$$

Q2. State without proof [**2**] Laurent's theorem for a function f holomorphic in an annular region $r < |z - a| < R$, and write down [**2**] the expression for the coefficients in the Laurent expansion as a contour integral.

For complex variables t, z and integer n, the function $J_n(z)$ is defined as the Laurent coefficient of t^n in the following Laurent expansion:

$$\exp\Big(\frac{1}{2}z\big(t - \frac{1}{t}\big)\Big) = \sum_{n=-\infty}^{\infty} t^n J_n(z).$$

Show that
(i) [**6**]

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - z\sin\theta)d\theta;$$

(ii) [**3**]

$$J_n(z) = \sum_{m=0}^{\infty} \frac{(-)^m(\frac{1}{2}z)^{n+2m}}{m!(n+m)!};$$

(iii) [**3**] for complex variables y, z,

$$J_n(y+z) = \sum_{m=-\infty}^{\infty} J_m(y)J_{n-m}(z);$$

(iv) [**2**] $J_{-n}(z) = (-)^n J_n(z)$;
(v) [**2**] $|J_n(z)| \le 1$ for z real.

Q3. (i) [**5**] Defining $\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}dt$, show that

$$\Gamma(x)\Gamma(1-x) = \int_0^\infty \frac{v^{x-1}}{1+v}dv \qquad (0 < x < 1).$$

(ii) [**5**] By integrating the many-valued function z^{a-1} round a suitable key-hole contour, or otherwise, show that

$$I := \int_0^\infty \frac{x^{a-1}}{1+x} dx = \pi / \sin \pi a \qquad (0 < a < 1).$$

(iii) [**3**] Hence or otherwise show that

$$\Gamma(x)\Gamma(1-x) = \pi / \sin \pi x \qquad (0 < x < 1).$$

(iv) [**3**] Deduce that for all complex z,

$$\Gamma(z)\Gamma(1-z) = \pi / \sin \pi z, \qquad \frac{1}{\Gamma(z)} \cdot \frac{1}{\Gamma(1-z)} = \frac{\sin \pi z}{\pi}.$$

(v) [**4**] Describe the behaviour of each term in each equation as a function of z.

Q4. (i) [**10**] Show that for $|a| < 1$,

$$I := \int_0^{2\pi} \frac{d\theta}{1 + a^2 - 2a \cos\ \theta} = \frac{2\pi}{1 - a^2}. \qquad [\mathbf{8}]$$

Find [**2**] the value when $|a| > 1$.
(ii) [**10**] Find

$$I := \int_0^\infty \frac{\cos x}{(a^2 + x^2)^2} dx \qquad (a > 0).$$

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