Q1. (i) Show that cos $n\theta$ is a polynomial T_n in $c := \cos\ \theta$. **[4]**
(ii) Find the leading coefficient of T_n. **[4]**
(iii) Consider the sequence $\cos(\pi/2^n)$, $n = 1, 2, \ldots$. Show, by induction or otherwise, that
(a) $\cos(\pi/2^n)$ can be obtained from integers by arithmetic operations and taking of square roots; **[4]**
(b) $\cos(\pi/2^n)$ is a zero of a polynomial P_n with integer coefficients (such a number is called an *algebraic number*). **[4]**
(c) Find the degree of P_n. **[4]**

Q2. (i) State without proof Cantor's theorem on a decreasing sequence of compact sets K_n. **[3]**
(ii) State and prove Cauchy's theorem for triangles. **[3, 14]**

Q3. (i) Defining $\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}dt$, show that

$$\Gamma(x)\Gamma(1-x) = \int_0^\infty \frac{v^{x-1}}{1+v}dv \qquad (0 < x < 1). \qquad \textbf{[5]}$$

(ii) By integrating the many-valued function $z^{a-1}/(1+z)$ round a suitable keyhole contour, or otherwise, show that

$$I := \int_0^\infty \frac{x^{a-1}}{1+x}dx = \pi/\sin\pi a \qquad (0 < a < 1). \qquad \textbf{[5]}$$

(iii) Hence or otherwise show that

$$\Gamma(x)\Gamma(1-x) = \pi/\sin\pi x \qquad (0 < x < 1). \qquad \textbf{[3]}$$

(iv) Deduce that for all complex z,

$$\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z, \qquad \frac{1}{\Gamma(z)}\cdot\frac{1}{\Gamma(1-z)} = \frac{\sin\pi z}{\pi}. \qquad \textbf{[3]}$$

(v) Describe the behaviour of each term in each equation as a function of z. **[4]**

Q4. (i) Show (by using a sector contour, or a keyhole contour, or otherwise) that

$$I_n := \int_0^\infty \frac{1}{1+x^n}\,dx = \frac{\pi}{n\sin\pi/n} \qquad (n = 2, 3, ...). \qquad \textbf{[10]}$$

(ii) Show that

$$\zeta(2) := \sum_{n=1}^\infty 1/n^2 = \pi^2/6 \qquad \textbf{[10]}$$

(you may quote any results you need without proof, but should state them clearly).

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