M2PM3 COMPLEX ANALYSIS: EXAMINATION, 2008

Q1. (i) Evaluate $(1+2i)^2$. Hence or otherwise find both roots of the complex quadratic $z^2 + 2iz + 2 - 4i$, in the form $a + ib$ with a, b real. **[5]**
(ii) Find the four roots of the quartic $z^4 + 1$. **[2]**
Plot them in the Argand diagram. **[2]**
Factorize the quartic
(a) as a product of four complex linear factors, **[2]**
(b) as a product of two real quadratics. **[2]**
(iii) By considering the ellipse $x^2/a^2 + y^2/b^2 = 1$ $(a,\ b > 0)$ parametrized by $x = a\cos\theta$, $y = b\sin\theta$, or otherwise, show that

$$\int_0^{2\pi} \frac{d\theta}{a^2\cos^2\theta + b^2\sin^2\theta} = \frac{2\pi}{ab}. \qquad [\mathbf{7}]$$

Q2. Let $f(z) = u(x,y) + iv(x,y)$ be holomorphic in a domain D.
(i) State without proof the Cauchy-Riemann equations for u, v. **[2]**
(ii) Define the term harmonic, and show that u, v are harmonic. **[4]**
(iii) Given u, describe how to find v and f. **[4]**
(iv) If $u(x,y) = x^3 - 3xy^2$, find v and f. **[4]**
(v) If $u(x,y) = x/(x^2+y^2)$ and the domain D does not contain the origin, find v and f. **[4]**
(vi) Check without differentiation that the u in (iv) and (v) are indeed harmonic. **[2]**

Q3. (i) State without proof Cauchy's integral formula for the value $f(a)$ of a function f holomorphic at a point a inside a contour γ. **[1]**
(ii) Show that

$$f'(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{(z-a)^2}. \qquad [\mathbf{10}]$$

(iii) State without proof the corresponding formula for $f^{(n)}(a)$. **[2]**
(iv) If γ is the circle centre a and radius R, and $|f(z)| \le M$ for $|z-a| \le R$, show that

$$|f^{(n)}(a)| \le n!M/R^n. \qquad [\mathbf{2}]$$

(v) What is meant by saying that f is an entire function? **[1]**
(vi) If f is entire and $|f(z)| \le c|z|^k$ for $|z|$ large and some constant c, show that f is a polynomial of degree at most k. **[4]**

Q4. (i) Show (by considering the unit circle parametrized by $z = e^{i\theta}$, or otherwise) that

$$\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}d\theta = \frac{\pi}{12}, \qquad \int_0^{2\pi} \frac{\sin 3\theta}{5-4\cos\theta}d\theta = 0. \qquad [\mathbf{10}]$$

(ii) Show that

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{2}. \qquad [\mathbf{10}]$$