M2PM3 EXAMINATION SOLUTIONS 2009

Q1. (i)

(a). $|z| = 2$; $z = 2(\frac{1}{2} + i\frac{\sqrt{3}}{2}) = 2(\cos\ \pi/3 + i \sin\ \pi/3)$: $z = 2e^{i\pi/3}$. **[1]**

(b) $z^5 = 2^5 e^{5i\pi/3} = 32e^{5i\pi/3}$ or $32e^{-i\pi/3}$ (polar); **[1]**

$z^5 = 32(\frac{1}{2} - i\frac{\sqrt{3}}{2}) = 16(1 - i\sqrt{3})$ (cartesian). **[1]**

(c) $z^{-1} = \frac{1}{2}e^{-i\pi/3}$ (polar); **[1]**

$$z^{-1} = \frac{1}{1 + i\sqrt{3}} = \frac{1 - i\sqrt{3}}{1 + 3} = \frac{1}{4} - i\frac{\sqrt{3}}{4} \qquad \text{(cartesian).} \qquad ([\mathbf{1}])$$

(ii) The singularities of $1/(1 + \cosh\ z)$ are the zeros of $1 + \cosh\ z$, the points where

$$\cosh\ z = \frac{1}{2}(e^z + e^{-z}) = -1, \qquad e^{2z} + 2e^z + 1 = 0, \qquad (e^z + 1)^2 = 0,$$

$$e^z = -1 = e^{(2n+1)i\pi}, \qquad z = (2n + 1)i\pi. \qquad ([\mathbf{4}])$$

Near such a point, take $z = (2n + 1)i\pi + \zeta$, ζ small.

$$1 + \cosh\ z = 1 + \frac{1}{2}(e^{(2n+1)i\pi}e^{\zeta} + e^{-(2n+1)i\pi}e^{-\zeta}) = 1 - \frac{1}{2}(e^{\zeta} + e^{-\zeta}).$$

Expanding the two exponential series, the constant terms go (they must, at a zero), and the ζ terms, leaving

$$1 + \cosh\ z = -\frac{1}{2}\zeta^2 + O(\zeta^3).$$

So $1 + \cosh\ z$ has a double zero at $(2n + 1)i\pi$, so $1/(1 + \cosh\ z)$ has a double pole there. (This also follows from $f(z) = e^z/(1 + e^z)^2$.) **[4]**

(iii) $f(z) = g(z)/(z - a)^k$, where g is holomorphic and $g(a) \neq 0$. **[3]**

So

$$f' = \frac{(z - a)^k g' - g.k(z - a)^{k-1}}{(z - a)^{2k}} = \frac{(z - a)g' - kg}{(z - a)^{k+1}}.$$

The numerator is $-kg(a) \neq 0$ at $z = a$. So f' has a pole at a of order $k + 1$, as required. **[4]**

Alternatively, the Laurent expansion of f at a has the form

$$f(z) = c_{-k}(z - a)^{-k} + \sum_{n=-k+1}^{\infty} c_n(z - a)^n, \qquad c_{-k} \neq 0.$$

We can differentiate this power series term by term, giving

$$f'(z) = -kc_{-k}(z - a)^{-k-1} + \sum_{n=-k+1}^{\infty} nc_n(z - a)^{n-1},$$

which shows that f' has a pole of order $k + 1$ at a.

Q2. (i) D is a *star-domain* with *star-centre* $z_0 \in D$ if for each $z \in D$ the line-egment $[z_0, z] \subset D$. **[2]**

(ii) *Proof of the Theorem of the Primitive.* Take any $z_1 \in D$. We prove that $F'(z_1)$ exists and is $f(z_1)$. As D is open and $z_1 \in D$, some neighbourhood $N(z_1, \epsilon_1) \subset D$. For $|h| < \epsilon_1$, $z_1 + h \in N(z_1, \epsilon_1) \subset D$. So as $z_0, z_1 + h \in D$, the line-segment $[z_0, z_1 + h] \subset D$ (D star-shaped with star-centre z_0). Let γ be the triangle with vertices $z_0, z_1, z_1 + h$, Δ be the union of γ and its interior. Then as D is star-shaped, $\Delta \subset D$. **[3]**

By Cauchy's Theorem for Triangles,

$$\int_\gamma f = 0, \qquad \text{i.e.} \qquad \int_{[z_0,z_1]} f + \int_{[z_1,z_1+h]} f + \int_{[z_1+h,z_0]} f = 0.$$

The first term is $F(z_1)$; the third term is $-F(z_1 + h)$. So

$$F(z_1) + \int_{[z_1,z_1+h]} f - F(z_1 + h) = 0 : \qquad \frac{F(z_1 + h) - F(z_1)}{h} = \frac{1}{h}\int_{[z_1,z_1+h]} f.$$

([**3**])

As f is continuous (it is holomorphic!), $\forall \epsilon > 0\ \exists \delta > 0$ s.t. $|z - z_1| < \delta$ implies $|f(z) - f(z_1)| < \epsilon$. For z on the line-segment $[z_1, z_1 + h]$, $|h| < \delta$ implies $|z - z_1| < \delta$, so $|f(z) - f(z_1)| < \epsilon$. So by (ML),

$$|\frac{1}{h}\int_{[z_1,z_1+h]} \frac{f(z) - f(z_1)}{h} dz| \leq \frac{1}{|h|}.|h|\epsilon = \epsilon. \qquad ([\mathbf{3}])$$

Now

$$\frac{F(z_1 + h) - F(z_1)}{h} - f(z_1) = \frac{1}{h}\int_{[z_1,z_1+h]} (f(z) - f(z_1))dz.$$

By above, the RHS has modulus $< \epsilon$. So the LHS has modulus $< \epsilon$, which says that $F'(z_1)$ exists and is $f(z_1)$, as required. **[3]**

(iii) (a) D is the complex plane with a cut along the negative real axis. For, the formula fails to give a convergent integral for $z = 0$, as the real integral $\int_0^1 dx/x$ diverges. So 0 and all points on the far side of 1 from it, i.e. the negative real axis, are not in D. But any point z not in D can be joined to 1 by a line-segment avoiding the singularity at 0; on $[1, z]$ the integrand $1/w$ is continuous, so the integral $\int_{[1,z]} dw/w$ is convergent. **[3]**

(b) $h'(z) = f'(g(z))g'(z)$ (chain rule) $= g'(z)/g(z)$ ($f'(z) = 1/z$, by the Theorem of the Primitive) $= g(z)/g(z) = 1$ ($g' = g$ for the exponential function g). So $h'(z) = 1$, $h(z) = z$ as required. **[3]**

(Interpretation: f is the inverse function of the exponential function g, i.e. the logarithm. The cut serves to make the logarithm single-valued.)

Note. If f is holomorphic in a neighbourhood of some point z, then f is differentiable in some disc containing z, and discs are star-shaped. So the Theorem of the Primitive will always apply *locally* – if we keep the domain small enough. The danger is that if the domain becomes too big, the primitive ceases to be single-valued (and so no longer counts as a function) – as with the logarithm in this case, if the cut is omitted.

Q3. *Cauchy-Taylor Theorem.* If f is holomorphic in $N(a, R)$, then

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \qquad (z \in N(a,R)), \qquad ([\mathbf{2}])$$

where for $r \in (0, R)$ and γ the circle with centre a and radius r,

$$c_n = f^{(n)}(a)/n! = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}} dz. \qquad ([\mathbf{2}])$$

Proof. Choose $z \in N(a, R)$, and then choose r with $|z - a| < r < R$; take $\gamma = \gamma(a, r)$. By the Cauchy Integral Formula,

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} dw. \qquad ([\mathbf{2}])$$

Now

$$w - z = (w-a)\Big(1 - \frac{z-a}{w-a}\Big): \qquad \frac{1}{w-z} = \sum_{n=0}^{\infty} (z-a)^n/(w-a)^{n+1}. \quad ([\mathbf{4}])$$

Substitute and use uniformly convergent on compact subsets of $N(a, R)$ to interchange integration and summation:

$$f(z) = \sum_{n=0}^{\infty} (z-a)^n . \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^{n+1}} dw.$$

So

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n, \qquad \text{where} \qquad c_n = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^{n+1}} dw = f^{(n)}(a)/n!, \qquad ([\mathbf{4}])$$

by the Cauchy integral formula for the nth derivative, $CIF(n)$, as required.

Take $f(z) = (1 + z)^a$. The complex power $z^a = \exp(a \log\ z)$ has a branch-point at 0 (as the logarithm does). So $(1 + z)^a$ has a branch-point at -1. So the largest disc centre 0 in which it is holomorphic is $D = \{z : |z| < 1\}$. $f'(z) = a(1 + z)^{a-1}$, $f''(z) = a(a - 1)(1 + z)^{a-2}$, ... $f^{(n)}(z) = a(a - 1) \ldots (a - n + 1)(1 + z)^{a-n}$. So $f^{(n)}(0)/n! = a(a - 1) \ldots (a - n + 1)/n! = \binom{a}{n}$. By the Cauchy-Taylor theorem,

$$(1+z)^a = \sum_{n=0}^{\infty} \binom{a}{n} z^n, \qquad |z| < 1. \qquad ([\mathbf{4}])$$

(Newton's General Binomial Theorem). The ratio of consecutive terms on the right is

$$\frac{a(a-1)\ldots(a-n)z^{n+1}}{(n+1)!} \Big/ \frac{a(a-1)\ldots(a-n+1)z^n}{n!} = (a-n)z/(n+1) \to -z \qquad (n \to \infty).$$

By the Ratio Test, this converges where $|-z| < 1$, i.e. where $|z| < 1$, showing directly that the power series on the right has radius of convergence 1. **[2]**

Q4. (i) $f(z) := \sin\ z/(z^2+1)$ is holomorphic except at the roots of $z^2+1=0$, i.e. $z = \pm i$. These are both inside γ. **[2]**
By the Cover-Up Rule, as $f(z) = \sin\ z/(z-i)(z+i)$,

$$Res_i f = \sin\ i/(2i), \qquad Res_{-i} f = \sin -i/(-2i) = Res_i f. \qquad ([\mathbf{2}])$$

But by Euler's formula $sin\ z = (e^{iz} - e^{-iz})/2i$, so $\sin\ i = (e^{-1} - e)/2i = i(e - e^{-1})/2$, $\sum Res f = 2\sin\ i/(2i) = (e - e^{-1})/2$. **[2]**
So by Cauchy's Residue Theorem,

$$\int_\gamma f = 2\pi i \sum Res f = 2\pi i.(e-e^{-1})/2 = \pi i(e - e^{-1}). \qquad ([\mathbf{2}])$$

(ii) Use $f(z) = (e^{ipz} - e^{iqz})/z^2$. This has a pole at 0 (apparently double, but actually single: the numerator has a simple zero at 0). We use a semicircular contour, indented to avoid this pole – a contour Γ consisting of:
(i) Γ_1, the line segment $[-R, -r]$ (R large, $r > 0$ small);
(ii) Γ_2, the semi-circle centre 0 radius r in the upper half-plane, clockwise (negative sense);
(iii) $\Gamma_3 = [r, R]$;
(iv) Γ_4, the semi-circle centre 0 radius R in the upper half-plane, anticlockwise (positive sense). **[4]**
On Γ_4, $|e^{ipz}| = |e^{ip(x+iy)}| = e^{-py} \le 1$, as $p \ge 0$ and $y \ge 0$ in the upper half-plane, and similarly $|e^{iqz}| \le 1$. So $|f(z)| = O(1/R^2)$, and by (ML),
$\int_{\Gamma_4} f = O(1/R^2).\pi R = O(1/R) \to 0$ as $R \to \infty$.
As $R \to \infty, r \to 0$, $\int_{\Gamma_1} f + \int_{\Gamma_3} f \to I$, the required integral.
On Γ_2, $z = re^{i\theta}$,

$$f(z) = [(1+ipz-p^2z^2/2...)-(1+iqz-q^2z^2/2...)]/z^2 = [i(p-q)-\frac{1}{2}(p^2-q^2)z+...]/z,$$

$dz/z = ire^{i\theta}d\theta/re^{i\theta} = id\theta$, where θ goes from π to 0. So (changing the sign to interchange the limits of integration)

$$\int_{\Gamma_2} f = -\int_0^\pi [i(p-q) + O(r)](id\theta) \to \pi(p-q) \qquad (r \to 0).$$

Since $\int_\Gamma f = 0$ by Cauchy's Theorem, this gives $I + \pi(p-q) = 0$:
$I = -\pi(p-q) = \pi(q-p)$. **[4]**.

Take $p = 0$, $q = 2$. Since $1 - \cos\ 2x = 2\sin^2 x$, this gives

$$2\int_{-\infty}^{\infty} \Big(\frac{\sin\ x}{x}\Big)^2 dx = 2\pi,$$

or as the integrand is even,

$$\int_0^\infty \Big(\frac{\sin\ x}{x}\Big)^2 dx = \pi/2. \qquad ([\mathbf{4}])$$

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