M2PM3 EXAMINATION SOLUTIONS 2010

Q1. (i) [**9**]
(a) By de Moivre's Theorem,

$$\cos\ n\theta + i\sin\ n\theta = e^{in\theta} = (e^{i\theta})^n = (c+is)^n$$

$$= c^n + i\binom{n}{1}c^{n-1}.s - \binom{n}{2}c^{n-2}.s^2 - i\binom{n}{3}c^{n-3}.s^3 + \binom{n}{4}c^{n-4}.s^4 \dots$$

Taking real and imaginary parts:

$$\cos\ n\theta = c^n - \binom{n}{2}c^{n-2}s^2 + \binom{n}{4}c^{n-4}s^4 \dots, \qquad [\mathbf{3}]$$

$$\sin\ n\theta = \binom{n}{1}c^{n-1}s - \binom{n}{3}c^{n-3}s^3 \dots. \qquad [\mathbf{3}]$$

(b) Divide (by c^n top and bottom on the right):

$$\tan\ n\theta = \frac{\binom{n}{1}t - \binom{n}{3}t^3 + \binom{n}{5}t^5 \dots}{1 - \binom{n}{2}t^2 + \binom{n}{4}t^4 \dots}. \qquad [\mathbf{3}]$$

(ii) [**4**]
Take $n = 7$: $\tan 7\theta = 0$ iff $\sin 7\theta = 0$ iff

$$t[7 - \binom{7}{3}t^2 + \binom{7}{5}t^4 - t^6] = 0. \qquad [\mathbf{1}]$$

Now $\tan 7\theta = 0$ iff $7\theta = n\pi$, n integer, $\theta = n\pi/7$. There are 7 roots ($\tan 7\theta = 0$ iff $\sin 7\theta = 0$; by (i), this is a polynomial equation of degree 7; by the Fundamental Theorem of Algebra, this has 7 roots). [**1**]
Taking $n = 0$ gives $t = 0$, $\theta = 0$. Taking $n = 1, 2, \dots, 6$ gives the other 6 roots as the roots of the polynomial of degree 6 above in [.]. [**2**]
(iii) [**7**]
With γ the ellipse $x^2/a^2 + y^2/b^2 = 1$ parametrized by $x = a\cos\theta$, $y = b\sin\theta$, $\int_\gamma dz/z = 2\pi i$ by Cauchy's Residue Theorem, since $1/z$ has residue 1 at 0. [**2**]
So as $z = a\cos\theta + ib\sin\theta$ gives $dz = (-a\sin\theta + ib\cos\theta)d\theta$,

$$2\pi i = \int_0^{2\pi} \frac{-a\sin\theta + ib\cos\theta}{a\cos\theta + ib\sin\theta} d\theta \qquad [\mathbf{2}]$$

$$= \int_0^{2\pi} \frac{(-a\sin\theta + ib\cos\theta)(a\cos\theta - ib\sin\theta)}{a^2\cos^2\theta + b^2\sin^2\theta} d\theta$$

$$= \int_0^{2\pi} \frac{(b^2 - a^2)\sin\theta\cos\theta + iab(\cos^2\theta + \sin^2\theta)}{a^2\cos^2\theta + b^2\sin^2\theta} d\theta.$$

Equating imaginary parts,

$$\int_0^{2\pi} \frac{ab}{a^2\cos^2\theta + b^2\sin^2\theta} d\theta = 2\pi,$$

whence the result on dividing by ab. [**3**]
Parts (i), (ii) unseen; (iii) seen.

Q2. *Laurent's theorem*: If f is holomorphic in the annular region $r < |z-a| < R$, then f possesses an expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n \quad (r < |z-a| < R), \quad c_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{(z-a)^{n+1}}, \; [\mathbf{2+2}]$$

where γ is any positively oriented contour in the annulus with winding number 1 about a.
(i) [**6**]. Take γ the unit circle. Write the generating function as $f(t) = \exp(z(t-1/t)/2)$, for fixed z. Then Laurent's formula for c_n gives

$$J_n(z) = \frac{1}{2\pi i}\int_\gamma \exp\{z(w-1/w)/2\}dw/w^{n+1}. \qquad [\mathbf{2}]$$

With $w = e^{i\theta}$,

$$\begin{aligned} J_n(z) &= \frac{1}{2\pi}\int_0^{2\pi} \exp\{z(e^{i\theta} - e^{-i\theta})/2\}.e^{-(n+1)i\theta}.e^{i\theta}d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \exp\{iz(e^{i\theta} - e^{-i\theta})/2i\}.e^{-ni\theta}d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \exp\{-ni\theta + iz\sin\theta\}d\theta. \qquad [\mathbf{2}] \end{aligned}$$

The RHS is $\int_{-\pi}^{\pi} = \int_{-\pi}^{0} + \int_0^{\pi}$. In the first, replace θ by $-\theta$:

$$J_n(z) = \frac{1}{2\pi}[\int_0^\pi \exp\{in\theta - iz\sin\theta\}d\theta + \int_0^\pi \exp\{-in\theta + iz\sin\theta\}d\theta] :$$

$$J_n(z) = \frac{1}{\pi}[\int_0^\pi \cos\{n\theta - z\sin\theta\}d\theta]. \qquad [\mathbf{2}]$$

(ii) [**3**]

$$\exp(\frac{1}{2}z(t-1/t)) = \big(\sum_{r=0}^{\infty}(z/2)^r t^r/r!\big).\big(\sum_{s=0}^{\infty}(-)^s(z/2)^s t^{-s}/s!\big) = \sum_{n=-\infty}^{\infty} t^n \sum_{r-s=n} (-)^s (z/2)^{r+s}/r!s!,$$

the rearrangements being justified by the absolute convergence of the exponential series. By uniqueness of Laurent expansions, the coefficient of t^n on RHS is $J_n(z)$. The result follows on replacing s by m, r by $s+n = m+n$.
(iii) [**3**] $exp(\frac{1}{2}(y+z)(t-1/t))$ is both $\sum_n t^n J_n(y+z)$ and $exp(\frac{1}{2}y(t-1/t)).exp(\frac{1}{2}z(t-1/t)) = \sum_r t^r J_r(y).\sum_s t^s J_s(z)$, which is $\sum_n t^n \sum_{r+s=n} J_r(y)J_s(z)$. Write r, s as m, $n-m$ and equate coefficients: $J_n(y+z) = \sum_m J_m(y)J_{n-m}(z)$.
(iv) [**2**] This follows from the defining generating function on replacing t by $-t$.
(v) [**2**] This follows from (i) and $|\cos\,.| \le 1$ for real arguments.
Laurent's theorem: seen, lectures. Problem (Bessel coefficients): (ii) seen, Problems.

Q3 (*The Gamma function*). (i) [**5**] For $0 < x < 1$, the integrals for both $\Gamma(x)$ and $\Gamma(1-x)$ converge, and $\Gamma(x)\Gamma(1-x) = \int_0^\infty t^{x-1}e^{-t}dt. \int_0^\infty u^{-x}e^{-u}du$. Substituting $u = tv$, the second integral on the right is $t^{1-x}\int_0^\infty v^{-x}e^{-tv}dv$. Cancelling powers of t and changing the order of integration, the RHS becomes

$$\int_0^\infty v^{-x}dv. \int_0^\infty e^{-(1+v)t}dt \quad = \int_0^\infty v^{-x}.\frac{1}{1+v}dv. \int_0^\infty e^{-w}dw \quad (w := (1+v)t).$$

The w-integral is 1. Interchanging x and $1-x$ (which preserves the LHS, and so the RHS also) gives

$$\Gamma(x)\Gamma(1-x) = \int_0^\infty \frac{v^{x-1}}{1+v}dv. \qquad [\mathbf{5}]$$

(ii) [**5**] *Cut* the plane along the positive real axis. Take $f(z) := z^{a-1}/((1+z)$; this is now holomorphic, except for a simple pole at $z = -1 = e^{i\pi}$, with residue (by the cover-up rule) $e^{i\pi(a-1)}$. Take γ the keyhole contour consisting of:
γ_1, the top of the x-axis between ϵ and R;
γ_2, a large circle radius R, +ve sense [avoiding the cut];
γ_3, the bottom of the x-axis from R to ϵ;
γ_4, a small circle around the origin of radius ϵ, -ve sense [avoiding the cut].
$\int_\gamma f = \sum_1^4 \int_{\gamma_i} f = \sum_1^4 I_i$, say.
$I_1 \to I$ as $R \to \infty$, $\epsilon \to 0$.
By ML, $I_2 = O((R^{a-1}/R).R) = O(R^{a-1}) \to 0$ as $R \to \infty$, as $a < 1$.
On γ_3, $z = xe^{2\pi i}$, $z^{a-1} = x^{a-1}e^{2\pi i(a-1)}$, so $I_3 \to -e^{2\pi i(a-1)}.I = -e^{2\pi ia}I$.
By ML, $I_4 = O(\epsilon^{a-1}.\epsilon) = O(\epsilon^a) \to 0$ as $\epsilon \to 0$, as $a > 0$.
By Cauchy's Residue Theorem, this gives

$$I(1 - e^{2\pi i(a-1)}) = 2\pi i.e^{\pi i(a-1)} = -2\pi ie^{i\pi a}:$$

$$I = -2\pi ie^{\pi ia}/(1 - e^{2\pi ia}) = -\pi.2i/(e^{-\pi ia} - e^{\pi ia}) = -\pi/-\sin\pi a = \pi/\sin\pi a.$$

(iii) [**3**] Changing a, x in (ii) to x, v and combining with (i) gives

$$\Gamma(x)\Gamma(1-x) = \pi/\sin\pi x \qquad (0 < x < 1).$$

(iv) [**3**] Since $\Gamma(z)\Gamma(1-z) - \pi/\sin\pi z$ vanishes on the interval $(0,1)$, by above, and this set has a limit point in the region (the plane less the integers) in which the function is holomorphic, it vanishes identically by *analytic continuation)*. So for all complex z,

$$\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z, \qquad \frac{1}{\Gamma(z)}.\frac{1}{\Gamma(1-z)} = \frac{\sin\pi z}{\pi}.$$

(v) [**4**] The Gamma function $\Gamma(z)$ has poles at 0 and the negative integers. So $\Gamma(1-z)$ has poles at the positive integers, and $\Gamma(z)\Gamma(1-z)$ has poles at the integers. So too does $\pi/\sin\ \pi z$. But $\Gamma(z)$ has no zeros (or $\sin\pi z$ would have a pole). So: $\Gamma(z)$, $\Gamma(1-z)$, $\pi/\sin\ \pi z$ have poles but no zeros; $1/\Gamma(z)$, $1/\Gamma(1-z)$, $(\sin\ \pi z)/\pi$ are *entire* – they have (integer) zeros but no poles.
All seen – (v) re analytic continuation in Ch. II, (ii) re integration in Ch. III.

Q4. (i) [**10**] Write cos $\theta = (e^{i\theta} + e^{-i\theta})/2$, and then $z := e^{i\theta}$, γ for the unit circle. Then $dz = ie^{i\theta}d\theta$, $d\theta = dz/(iz)$,

$$I = \int_\gamma \frac{dz}{iz[1 - a(z + 1/z) + a^2]} = i\int_\gamma \frac{dz}{[az^2 - a^2z - z + a]}$$

$$= i\int_\gamma \frac{dz}{(az-1)(z-a)} = \frac{i}{a}\int_\gamma \frac{dz}{(z-1/a)(z-a)}. \qquad [\mathbf{4}]$$

So the integrand has simple poles at a, $1/a$, only the first being inside γ. The residue at a of $1/((z-a)(z-1/a))$ is $1/(a-1/a) = a/(a^2-1) = -a/(1-a^2)$, by the Cover-Up Rule. So by Cauchy's Residue Theorem,

$$I = \frac{i}{a}.2\pi i.\frac{-a}{1-a^2}: \qquad I = \frac{2\pi}{1-a^2}. \qquad [\mathbf{4}]$$

For $|a| > 1$, the pole inside γ is now $1/a$, with residue $1/(1/a - a) = a/(1-a^2) = -a/(a^2-1)$. The argument above now gives

$$I = \frac{2\pi}{a^2-1}. \qquad [\mathbf{2}]$$

(ii) [**10**] Take $f(z) := e^{iz}/(a^2+z^2)^2$ and γ the semicircle γ_1 in the upper half-plane with base $\gamma_2 := [-R, R]$. [**2**]
On $|z| = R$, $y = Im\ z \geq 0$, $f(z) = e^{ix}.e^{-y}/(a^2+z^2)^2$, $|f| = O(1/R^4)$, so by ML $\int_{\gamma_1} f = O(R/R^4) = O(1/R^3) \to 0$ as $R \to \infty$, [**1**]
while

$$\int_{\gamma_2} f \to \int_{-\infty}^{\infty} \frac{e^{ix}}{(a^2+x^2)^2}dx = \int_{-\infty}^{\infty} \frac{\cos\ x}{(a^2+x^2)^2}dx = 2\int_0^{\infty} \frac{\cos\ x}{(a^2+x^2)^2}dx = 2I.$$

[**1**]

f has double poles at $\pm ia$, of which only ia is inside γ. [**1**]
For $z = ia + \zeta$, ζ small,

$$f(z) = \frac{e^{i(ia+\zeta)}}{(a^2 + (-a^2 + 2ia\zeta + \zeta^2))^2} = e^{-a}.e^{i\zeta}.\zeta^{-2}.(2ia)^{-2}.(1+\zeta/2ia)^{-2}$$

$$= \frac{e^{-a}}{-4a^2}.\zeta^{-2}.(1 + i\zeta + \ldots)(1 - \zeta/ia\ldots) = \frac{e^{-a}}{-4a^2}.\zeta^{-2}.(1 + i\zeta(1+1/a) + \ldots).$$

The coefficient of $1/\zeta$ on RHS is $Res_{ia}f = -i(1+1/a).e^{-a}/(4a^2)$. [**3**]
So by Cauchy's Residue Theorem, $2I = 2\pi i.[-i(1+1/a).e^{-a}/(4a^2)]$:

$$I = \frac{\pi(1+1/a)}{4a^2e^a}. \qquad [\mathbf{2}]$$

Both parts are unseen in this form, but are variants or extensions of examples done in lectures.

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