m2pm3infprod.tex

III.9. Infinite products for sin, cos *and* tan.

From III.8 (Lecture 32(11)):

$$cosec\ z = \frac{1}{z} + 2z\sum_{n=1}^{\infty}\frac{(-)^n}{z^2 - n^2\pi^2} = \frac{1}{z} + 2z\sum_{even} - 2z\sum_{odd}. \qquad (i)$$

Similarly,

$$cot\ z = \frac{1}{z} + 2z\sum_{n=1}^{\infty}\frac{1}{z^2 - n^2\pi^2}. \qquad (ii)$$

Now (with $D := d/dz$ the differentiation operator)

$$D\log\ \tan\frac{1}{2}z = cosec\ z, \qquad D\log\ \sin z = cot\ z,$$

$$D\log(1 - \frac{z^2}{n^2\pi^2}) = \frac{-2z/n^2\pi^2}{1 - z^2/n^2\pi^2} = \frac{2z}{z^2 - n^2\pi^2}.$$

Integrating (ii) gives (using Π for product, as we do $\sum$ for sum)

$$\log\sin z - \log z = \sum_1^{\infty}\log(1 - \frac{z^2}{n^2\pi^2}) = \log\Pi_1^{\infty}(1 - \frac{z^2}{n^2\pi^2}).$$

Taking exponentials,

$$\frac{\sin z}{z} = \Pi_1^{\infty}(1 - \frac{z^2}{n^2\pi^2}) \qquad (iii)$$

(both sides $\to 1$ as $z \to 0$, accounting for the constant of integration). Similarly, integrating (i) gives

$$\log\tan\frac{1}{2}z = \log z + \log c + \sum_{even}\log(1 - \frac{z^2}{n^2\pi^2}) - \sum_{odd}\log(1 - \frac{z^2}{n^2\pi^2}).$$

Take exponentials:

$$\tan\frac{1}{2}z = cz\Pi_{even}(1 - \frac{z^2}{n^2\pi^2})/\Pi_{odd}(1 - \frac{z^2}{n^2\pi^2}).$$

Both products $\to 1$ as $z \to 0$, so for small z, LHS $\sim \frac{1}{2}z$, RHS $\sim cz$: $c = 1/2$. Replace z by $2z$:

$$\tan z = \frac{\sin z}{\cos z} = z\Pi_{n=1}^{\infty}(1 - \frac{z^2}{n^2\pi^2})/\Pi_{n=1}^{\infty}(1 - \frac{4z^2}{(2n-1)^2\pi^2}) \qquad (iv)$$

(cancelling 4 in $(2z)^2/(2n)^2$). From (iii) and (iv),

$$cos z = \Pi_1^\infty (1 - \frac{4z^2}{(2n-1)^2\pi^2}). \qquad (v)$$

Note that the infinite products for sin and cos display zeros at the integers and the half-integers, as they should.

Taking $z = \pi/2$ in the product (iii) for sin:

$$\pi^{-1} = \frac{1}{2}\Pi_1^\infty (1 - \frac{1}{4n^2}).$$

This is *Wallis' product for* π (John WALLIS (1616-1703), *Arithmetica infinitorum*, 1656 – see Coursework 2 Q5 for Wallis' product by Real Analysis).

By (iii) and the power series for sin,

$$\frac{\sin z}{z} = \sum_{k=0}^\infty \frac{(-)^k z^{2k}}{(2k+1)!} = \Pi_1^\infty (1 - \frac{z^2}{n^2\pi^2}).$$

Equate coefficients of z^2:

$$-\frac{1}{3!} = -\frac{1}{\pi^2}.\sum_1^\infty \frac{1}{n^2}: \qquad \zeta(2) = \sum_1^\infty 1/n^2 = \pi^2/6,$$

again. Similarly, equating coefficients of z^4 gives

$$\frac{1}{5!} = \frac{1}{120} = \frac{1}{\pi^4}.\Sigma\Sigma_{1 \le r < s < \infty}\frac{1}{r^2 s^2}.$$

But

$$(\sum_{r=1}^\infty \frac{1}{r^2}).(\sum_{s=1}^\infty \frac{1}{s^2}) = \sum_{n=1}^\infty \frac{1}{n^4} + 2\Sigma\Sigma_{1 \le r < s < \infty}\frac{1}{r^2 s^2}.$$

The LHS is $\zeta(2)^2 = (\pi^2/6)^2 = \pi^4/36$. By above, the RHS is $\sum_1^\infty 1/n^4 + 2.\pi^4/120$. So

$$\zeta(4) := \sum_{n=1}^\infty \frac{1}{n^4} = \pi^4(\frac{1}{36} - \frac{1}{60}) = \frac{\pi^4}{360}(10-6) = 4\pi^4/360 = \pi^4/90,$$

again. The same method shows that $\zeta(6) := \sum_1^\infty 1/n^6$ is a rational multiple of π^6, etc.

We quote (Weierstrass' product for the Gamma function)

$$1/\Gamma(z) = ze^{\gamma z}\Pi_{n=1}^\infty\{(1 + \frac{z}{n})e^{-z/n}\}$$

(where γ is Euler's constant – this shows again that Γ has poles, $1/\Gamma$ has zeros, at $0, -1, -2, \dots$). From this and $\Gamma(z)\Gamma(1-z) = \pi/\sin\,\pi z$, we can recover the product (iii) for the sin (exercise).