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Lecture 1. 10.1.2011.

Chapter I. PRELIMINARIES

0. Why complex Analysis?

Complex Analysis appeared in 1545: *Ars Magna*, Girolano CARDANO (1501-1576) (all dates are in *Dramatis Personae: Who Did What When*, on the course website). Calculus was introduced by Newton & Leibniz in the 1670s. Taylor's Theorem is named after Brook TAYLOR (1685-1731), *Methodus Incrementorum*, 1715. Complex analysis proper was introduced by Augustin-Louis CAUCHY in 1825-29; the main result, Cauchy's Theorem, concerns complex integrals, hence complex integral calculus.

We learn integral calculus in two steps: first at school, with the 'Sixth Form integral', and then at university in Real Analysis, with the Riemann integral (G. F. B. RIEMANN (1826-66) in 1854 – essentially the Sixth Form integral in 'ϵ-δ language').

Complex Analysis is NEW. It is core material for any University Mathematics course, for two reasons:

1. It provides us with a very powerful technique for evaluating (real) integrals and summing (real) series. Sample results:

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2};$$

$$\int_{-\infty}^{\infty} e^{ixt}\frac{e^{-\frac{1}{2}t^2}}{\sqrt{2\pi}} = e^{-\frac{1}{2}t^2};$$

$$\int_{-\infty}^{\infty} \frac{e^{ixt}}{\pi(1+x^2)}\,dx = e^{-|t|}$$

[characteristic function (CF), or Fourier transform, of the standard normal distribution and the Cauchy distribution respectively].

We can also find the limit of interesting sums, e.g.

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6};$$

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

2. Complex Analysis is simpler than Real Analysis.

E.g., take the Taylor Series for $f(x) = \exp\{-1/x^2\}$, x real, at $x = 0$ (with $f(0)$ defined as 0, since $f(x) \to 0$ as $x \to 0$ – indeed, very fast). We shall see that $f^{(n)}(x)$ exists at $x = 0$ for all $n = 0, 1, ...$ and is 0. So the Taylor Series of $f(x)$ at $x = 0$ is $\sum_{n=0}^{\infty} 0.x^n/n!$ – converges to 0. But we expect the Taylor series of a function f *to converge to the function.* So this example – convergence of the Taylor series to 'the wrong function' – seems pathological.

But for z complex, $f(z) = \exp\{-1/z^2\}$ behaves very badly at $z = 0$, as $f(yi) = \exp\{+1/y^2\} \to \infty$ as $y \to 0$ (again, very fast). So this pathological behaviour is no longer surprising: 0 is a point of extreme bad behaviour of f (in II.9 we shall classify points of bad behaviour – *singularities*; this is the worst kind – an *essential singularity*).

To understand Taylor Series (= power series) we need a complex viewpoint. Complex Analysis *is* the study of power series, from one point of view (Cauchy-Taylor Theorem: II.7).

Complex Analysis brings powerful new ideas to bear that have no counterpart in Real Analysis. For example, in Real Analysis, knowing a function on one part of the real line gives no information about its values on other parts. By contrast, in Complex Analysis, we shall see that knowing the values of a (holomorphic) function in an infinite set with a limit point in a region of good behaviour – say, an open disc, however small, or an open interval – gives (in principle) knowledge of the function *everywhere* (everywhere that it can be defined, that is). This is the basis of the powerful idea of *analytic continuation* (II.8). For example, we will show (Euler's reflection formula)

$$\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x \qquad (0 < x < 1).$$

From this, we will obtain

$$\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z \qquad (z \in \mathbf{C}).$$

This is surprising! The open interval $(0, 1)$ on the real line is a tiny subset of the complex plane. Nevertheless, if the formula holds on $(0, 1)$, it holds everywhere. Similarly, we will find that the values of a (holomorphic) function *on* a contour (a circle, say) determine the values *inside* it (how?!). Nothing like this is possible in Real Analysis! The point of such examples is to show that Complex Analysis is a genuinely new subject – not at all like the Analysis of Semesters 1-3 – and is extremely powerful. This is why it is a core course, in this or any other undergraduate Mathematics curriculum.