m2pm3l11.tex

Lecture 10. 31.1.2011.

Chapter II. Holomorphic (Analytic) Functions: Theory

1. Special Complex Functions.

1. *Polynomials*. $f(z) = a_0 + a_1 z + ... + a_n z^n$ $(a_i \in \mathbf{C}, a_n \neq 0)$. This is a (complex) *polynomial* of *degree* n. We shall prove (II.6, Fundamental Theorem of Algebra) that f has n roots.

2. *Exponentials*

$\exp(z) = \sum_{n=0}^{\infty} z^n/n!$. This function is absolutely and uniformly convergent on all closed discs in $\mathbf{C}$.

$$\exp(z_1 + z_2) = \sum_{n=0}^{\infty} \frac{(z_1+z_2)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} z_1^k z_2^{n-k} \binom{n}{k}.$$

Here $0 \le k < \infty$. Putting $l := n - k$: $0 \le k, l < \infty$, giving

$$\exp(z_1 + z_2) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{z_1^k}{k!} \times \frac{z_2^l}{l!} = \left(\sum_{k=0}^{\infty} \frac{z_1^k}{k!}\right) \left(\sum_{l=0}^{\infty} \frac{z_2^l}{l!}\right) = \exp(z_1) \times \exp(z_2)$$

(the rearrangement is justified by absolute convergence). That is,

$$\exp(z_1 + z_2) = \exp(z_1) \times \exp(z_2).$$

Recall $e = \exp(1) = \sum_{n=0}^{\infty} 1/n!$. Now:

$$\exp(nz) = [\exp(z)]^n \quad (n \in \mathbf{N}); \qquad \exp(z/m) = \exp(z)^{1/m} \quad (n \in \mathbf{N}).$$

Combining:

$$\exp(\frac{m}{n} z) = [\exp(z)]^{m/n} \quad (m, n \in \mathbf{N}); \qquad \exp(qz) = [\exp(z)]^q \quad (q \in \mathbf{Q}).$$

Taking $z = 1$ gives $\exp(q) = [\exp(1)]^q = e^q$ for $q \in \mathbf{Q}$. So $\exp(q) = e^q$ for $q \in \mathbf{Q}$. Hence

$$\exp(x) = e^x \qquad (x \in \mathbf{R})$$

(both sides are continuous: take q_n rational, $q_n \to x$.

3. *Trigonometric functions.* Recall from Chapter I:

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - ..., \qquad \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - ...,$$

$$\exp(iz) = \cos(z) + i\sin(z).$$

For $z = \theta$ real, $\exp(i\theta) = \cos\theta + i\sin\theta$ (Euler's formula, Chapter I).
We *define* $e^{i\theta}$ by $e^{i\theta} = \exp(i\theta) = \cos\theta + i\sin\theta$. Then for $z = x + iy$,

$$\exp(z) = \exp(x + iy) = \exp(x).\exp(iy) = e^x.e^{iy},$$

by above. We *define* e^{x+iy}, or e^z, as the RHS ("rule of indices", complex case). Then

$$\exp(z) = e^z \quad (z \in \mathbf{C}).$$

Henceforth, we use e^z for $\exp(z)$.
4. *Hyperbolic functions*

$$\cosh z = \frac{1}{2}(e^z + e^{-z}), \qquad \sinh z = \frac{1}{2}(e^z - e^{-z}), \qquad \tanh z = \frac{\sinh z}{\cosh z},$$

$$\frac{d}{dz}\cosh z = \sinh z, \qquad \frac{d}{dz}\cosh z = \cosh z, \qquad \cosh^2 z - \sinh^2 z = 1.$$

5. *Logarithms*

Recall that in Real Analysis, log is the inverse function of exp:

$$\log x = y \text{ means } e^y = x.$$

This extends to $\mathbf{C}$, as follows: for $z, w \in \mathbf{C}$,

$$\log z = w \text{ means } e^w = z.$$

But: $e^{2\pi i} = 1$, so $e^{2\pi ki} = 1 \ \forall k \in \mathbf{Z}$. So if $e^w = z$, also $e^{w+2\pi ki} = z$. So if $\log z = w$, also $\log z = w + 2\pi ki$: the log is *not* single-valued, and is determined only to within additive multiples of $2\pi i$. In particular, log is not a *function* as previously defined.

There are three ways to proceed:
(i) *Many-valued functions.*
We can regard log as a *many-valued function* (as with $\sin^{-1} = \arcsin$).
(ii) *Cuts.*
Cut the complex plane $\mathbf{C}$ by removing (e.g.) the negative real axis.