m2pm3l12(11).tex

Lecture 12. 3.2.2011.

So if f is differentiable at z_0 with derivativ e $f'(z_0)$,

$$\forall\epsilon > 0, \exists\delta > 0 \text{ s.t. } \forall z \text{ with } |z - z_0| < \delta, \quad \left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \epsilon$$

($arg(z-z_0)$ can be anything!). Write $z-z_0 = h = k+il$ (k, l real), $f = u+iv$ (u, v real): $f(z) = u(x,y) + iv(x,y)$.
1. h real ($l = 0$).

$$\frac{u(x_0 + k, y_0) - u(x_0, y_0)}{k} + i\frac{v(x_0 + k, y_0) - v(x_0, y_0)}{k} \to f'(z_0) \quad (k \to 0):$$

$$u_x(x_0, y_0) + iv_x(x_0, y_0) = f'(z_0), \quad \text{writing } u_x \text{ for } \partial u/\partial x.$$

2. h imaginary ($k = 0$).

$$\frac{u(x_0, y_0 + l) - u(x_0, y_0)}{l} + i\frac{v(x_0, y_0 + l) - v(x_0, y_0)}{il} \to f'(z_0) \quad (l \to 0):$$

$$-iu_y(x_0, y_0) + v_y(x_0, y_0) = f'(z_0), \quad \text{writing } u_y \text{ for } \partial u/\partial y.$$

Combining, at (x_0, y_0)

$$u_x = v_y, \quad v_x = -u_y.$$

These are called the Cauchy-Riemann Equations, C-R.
So differentiability at $(x_0, y_0) \Rightarrow$ C-R at (x_0, y_0): C-R are *necessary* for differentiability. They are *not sufficient.*
Example. $f(z) = \sqrt{|xy|}$ ($z = x + iy = re^{i\theta}$). So $f, u, v \equiv 0$ on both axes. So $u_x, u_y, v_x, v_y \equiv 0$ on both axes and C-R holds at $(0,0)$. But:

$$\frac{f(z) - f(0)}{z - 0} = \frac{\sqrt{|r\cos\theta \cdot r\sin\theta|}}{re^{i\theta}} = e^{-i\theta}\sqrt{|\cos\theta\sin\theta|}.$$

RHS depends on θ, i.e. *how* $z = re^{i\theta} \to 0$: f is *not* differentiable at 0.
But there is a *partial* converse:

Theorem. If $f = u + iv$ and the partial derivatives u_x, u_y, v_x, v_y exist *and are continuous* in a neighbourhood of z_0, and satisfy the C-R equations at z_0, then f is dfferentiable at z_0.

Proof. Take $h = k + i\ell$ so small that $z = z_0 + k$ is in the neighbourhood where partials are continuous; then

$$u(x_0+k, y_0+\ell)-u(x_0, y_0) = [u(x_0+k, y_0+\ell)-u(x_0, y_0+\ell)]+[u(x_0, y_0+\ell)-u(x_0, y_0)].$$

By the Mean Value Theorem (MVT): for some $\theta \in (0, 1)$,

$$\begin{aligned}[u(x_0 + k, y_0 + \ell) - u(x_0, y_0 + \ell)]/k &= u_x(x_0 + \theta k, y_0 + \ell) \\ &= u_x(x_0, y_0) + o(1) \quad \text{as } h \to 0.\end{aligned}$$

(here we use the o-notation for the error term:'$o(1)$ as $h \to 0$' means '$\to 0$ as $h \to 0$'), by continuity of the partial u_x. Similarly,

$$\begin{aligned}[u(x_0, y_0 + \ell) - u(x_0, y_0)]/\ell &= u_y(x_0, y_0 + \theta' \ell) \quad \text{for some } \theta' \in (0, 1) \\ &= u_y(x_0, y_0) + o(1) \quad \text{as } h \to 0.\end{aligned}$$

Combining:

$$u(x_0 + k, y_0 + \ell) - u(x_0, y_0) = ku_x(x_0, y_0) + \ell u_y(x_0, y_0) + o(h),$$

where '$o(h)$' means 'smaller order of magnitude then h as $h \to 0$.' This combines two error terms, $o(k)$ and $o(l)$, both $o(h)$ as $h^2 = k^2 + l^2$, so $|k| \leq |h|$, $|l| \leq |h|$. Similarly,

$$v(x_0 + k, y_0 + \ell) - v(x_0, y_0) = kv_x(x_0, y_0) + \ell v_y(x_0, y_0) + o(h).$$

So

$$\begin{aligned}f(z_0 + h) - f(z_0) &= [u(x_0 + k, y_0 + \ell) - u(x_0, y_0)] + i[v(x_0 + k, y_0 + \ell) - v(x_0, y_0)] \\ &= ku_x + \ell u_y + ikv_x + i\ell v_y + o(h).\end{aligned}$$