m2pm3l13(11).tex

**Lecture 13**. 7.2.2011.

Replace $u_y$, $v_y$ on RHS by $-v_x$, $u_x$, using the C-R equations. The RHS becomes

$$(k+i\ell)u_x + i(k+i\ell)v_x + o(h) = h(u_x + iv_x) + o(h),$$

by C-R again. Divide by h:

$$\begin{aligned} f(z_0+h) - f(z_0)/h &= u_x(x_0,y_0) + iv_x(x_0,y_0) + o(1) \\ &\to u_x(x_0,y_0) + iv_x(x_0,y_0) \qquad \text{as } h \to 0. \end{aligned}$$

So $f'(z_0)$ exists and $= u_x(x_0,y_0) + iv_x(x_0,y_0)$. //
*Note.* There are *three* other ways to write the RHS here, by C-R.
*Harmonic functions.*

With $u$, $v$ as above,

$$\begin{aligned} u_{xx} &= (v_y)_x && \text{(by Cauchy-Riemann)} \\ &= (v_x)_y && \text{(interchanging the order of partial differentiation)} \\ &= (-u_y)_y && \text{(by Cauchy-Riemann)}: \end{aligned}$$

$$u_{xx} + u_{yy} = 0, \quad \text{or} \quad \Delta u = 0,$$

where $\Delta$ is the two-dimensional Laplacian operator $\partial^2/\partial x^2 + \partial^2/\partial y^2$, and similarly $\Delta v = 0$. We say that $u$, $v$ are *harmonic functions.*
*Note.* The interchange of the order of partial differentiation here is justified by Clairault's Theorem, which needs *continuity* of the relevant partials. We shall see later (II.7) that we have this and more, but assume it for now.

*Gradient and Directional Derivative.*

The *gradient* of $u$, grad $u$ or $\nabla u$, is the 2-vector $\begin{pmatrix} u_x \\ u_y \end{pmatrix}$. The *directional derivative* $D_{\mathbf{u}}u$ of $u$ in the *direction* of the unit vector $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is

$$D_{\mathbf{u}}u(x,y) = \lim_{t\to 0} \frac{u(x+tu_1, y+tu_2) - u(x,y)}{t} \quad \text{(where this exists).}$$

By the *proof* of the Theorem above: if the partials of $u$ exist and are continuous,
(i) $D_{\mathbf{u}}u$ exists, $\forall \mathbf{u}$;
(ii) $D_{\mathbf{u}}u = u_1 u_x(x,y) + u_2 u_y(x,y)$: $D_{\mathbf{u}}u = \mathbf{u}.\nabla u$. For in the Proof, $h = k + il$; $u_1 \leftrightarrow k$, $u_2 \leftrightarrow l$; $h = k + il \leftrightarrow u_1 + iu_2 \leftrightarrow \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ ('$1 \leftrightarrow \mathbf{i}$, $i \leftrightarrow \mathbf{j}$'). So,

$$\text{If } \mathbf{u} \perp \nabla u, D_{\mathbf{u}}.u = 0.$$

The directional derivative is the rate of change of $u$ in direction $\mathbf{u}$. This is 0 along a tangent to the curve $u = const$. So

$$\text{grad } u \perp \text{(tangent to) } u = const.$$

By Cauchy-Riemann,

$$\nabla u.\nabla u = \begin{pmatrix} u_x \\ u_y \end{pmatrix} . \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} u_x \\ u_y \end{pmatrix} . \begin{pmatrix} -u_y \\ u_x \end{pmatrix} = -u_x u_y + u_x u_y = 0 :$$

$$\nabla u \perp \nabla v.$$

Combining: Curves $u$ const. and $v$ const. cut orthogonally.
*Electromagnetism (EM).*
$u$ const.: equipotential curves (const potential), $v$ const.: lines of force.
*Fluid Mechanics.*
$u$ const.: equipotentials, $v$ const.: lines of flow.
*Gravitational Potential* (OS Maps).
$u$ const.: *contours*, $v$ const.: lines of steepest ascent/decent.
*Harmonic Conjugates.*

Given $u$, to find $v$ and $f$. By C-R, $v_x = u_y$. Integrate w.r.t. $y$:

$$v = \int u_x(x,y)\, dy + F(x).$$

Differentiating w.r.t. $x$:

$$v_x = \frac{\partial}{\partial x}[\int u_x\, dy] + F^{'}(x).$$

By C-R, $v_x = -u_y$:

$$-u_y = \frac{\partial}{\partial x}[\int u_x\, dy] + F^{'}(x).$$

Hence $F^{'}$, $F$, $v$, $f = u + iv$. The function $v$ is called the *harmonic conjugate* of $u$, $v = \tilde{u}$, $f = u + iv \to f = u + i\tilde{u}$. Similarly, given $v$ we can find $u$.