m2pm3l14(11).tex
Lecture 14. 8.2.2011.

3. Connectedness.

The theory in Ch. II is about nice functions (differentiable ones: Section 2) on nice sets (non-empty, open, connected ones). Connectedness is a *topological property*, not specific to **C**. We meet it now rather than in I.2, as we did not need it there – here it is essential.

Defn. In a topological space (in particular a metric space, in particular $\mathbf{R}^d$ or **C**) an open set S is *disconnected* $\Leftrightarrow$ it is the union of two disjoint non-empty open sets. Otherwise S is *connected*.

Example:

In **R**, $(-1,0) \cup (0,1)$ is disconnected, but $(-1,1) = (-1,0) \cup \{0\} \cup (0,1)$ is connected.

This is the key illustrative example. The left and right open intervals here have a common end-point, but as this is missing and not in the set its absence serves to *disconnect* their union.

Connected Sets in **R**.

We quote: the connected sets on **R** are the *intervals*.

This shows that 'connected' as a technical term defined above is being used in a sense consistent with its use in ordinary language.

Open Sets in **R**.

We quote: S open in $\mathbf{R} \Leftrightarrow S$ is a finite union or countably union of open intervals.

Defn. A set S is *polygonally connected* if any two points in S can be joined by a polygonal [continuous piecewise-linear curve] entirely contained in S.

We quote: (i) In **C**, an open set S is connected $\Leftrightarrow$ it is polygonally connected.

(ii) W.l.o.g., we can take the line-segments of the polygonal path horizontal or vertical.

For Proof, see e.g. Ahlfors, p.56-57 (Chapter 2, Section 1.3).

For general (not necessarily open) sets, polygonal connectedness is too restrictive, and we replace polygonal paths by paths (see Section 4 below).

If $z_1, z_2 \in S$, write $z_1 \sim z_2$, (or $z_1 \sim_S z_2$) if z_1, z_2 can be joined by a path that is contained in S (the path can be taken polygonal if S is open). This is an *equivalence relation* (reflexive, symmetric, and transitive), so it decomposes S into (disjoint) equivalence classes, called the *connected components* of S.

S is connected $\Leftrightarrow$ it only has *one* connected component.

A connected set S is called *simply connected* $\Leftrightarrow$ S^c is connected ('no holes'). Call S:

doubly connected $\Leftrightarrow$ S^c has *two* connected components ('one hole' – e.g., an annulus);

triply connected $\Leftrightarrow$ S^c has *three* connected components ('two holes');

n-ply connected $\Leftrightarrow$ S^c has n connected components ('$n-1$ holes').

Three examples:

1. *Annulus*: $\{z : 1 < |z| < 2\}$ is doubly connected.
2. *Disc*: $\{z : |z| < 1\}$ is simply connected.
3. *Punctured disc*: $\{z : 0 < |z| < 1\}$ is doubly connected.

Note. When we meet Cauchy's Residue Theorem (II.7), we find that our functions f holomorphic on domains D have points of bad behaviour – *singularities*. Each singularity needs to be excluded from D (by making a 'hole'): all the action is at the singularities.

4. Paths, Line Integrals, Contours

Defn. A *curve* $\gamma : [a, b] \to \mathbf{C}$ is a C^1-function $\gamma : t \mapsto \gamma(t) = \gamma_1(t) + i\gamma_2(t)$. Call $\gamma(a)$, the *beginning point* or *start* of γ, $\gamma(b)$ the *end-point* or *end* of γ. If $\gamma : [a, b] \to G$, G open, $G \subset \mathbf{C}$, call γ a curve *in* G.

If $f : G \to \mathbf{C}$ is differentiable, $f \circ \gamma(t) \mapsto f(\gamma(t)) : [a, b] \to \mathbf{C}$, and $(f \circ \gamma)'(t) = f'(\gamma(t))\gamma'(t)$ (Chain Rule).

We often need to join curves 'end to end', allowing 'corners', where things are not smooth.

Defn. 1. A *path* γ is a finite set of curves, $\gamma = \{\gamma_1, ..., \gamma_n\}$ (where each γ_i is in C^1), s.t. the end-point of each γ_i is the start of γ_{i+1}.

2. A set $S \subset \mathbf{C}$ is *pathwise connected* if any two points of S can be joined by a path entirely contained in S. (Polygonally connected $\Rightarrow$ pathwise connected, as the joining polygonal path is a joining path).

One can extend the definition above of connected sets to general sets (using the induced topology). We quote that:

pathwise connected $\Rightarrow$ connected. The converse holds in $\mathbf{C}$ (but not in general).

Example. The unit circle C is pathwise connected, and connected. It is not polygonally connected (and it is closed, not open).