m2pm3l15(11).tex
Lecture 15. 10.2.2011.

In a path $\gamma = \{\gamma_1, ...\gamma_n\}$ with γ_i *parametrised* by $[a_i, b_i]$, we may take

$$a = a_1 < b_1 = a_2 < b_2 = a_3 < ... < b_{n-1} = a_n < b_n = b.$$

Then $t \mapsto \gamma(t)$ is C^1, *except* at finitely many points $t = a_{i+1} = b_i$.

Defn. The *path integral*, or *line integral*, $\int_\gamma f$, is

$$\int_\gamma f, \text{ or } \int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma^{'}(t)\,dt := \sum_{i=1}^n \int_{a_i}^{b_i} f(\gamma(t))\gamma^{'}(t)\,dt.$$

Curve Length.
The *length* of a C^1 curve γ on $[a, b]$ is

$$L(\gamma) = \int_a^b \sqrt{\dot{\gamma}_1^2(t) + \dot{\gamma}_2^2(t)}\,dt \text{ or } \int_a^b |\dot{\gamma}^2(t)|\,dt.$$

The integrals above are all *Riemann integrals.*

Defn. 1. A path $\gamma : [a, b] \to \mathbf{C}$ is *closed* if $\gamma(b) = \gamma(a)$ (the two end-points of the curve are the same).
2. The path γ is *simple* if $\gamma(s) = \gamma(t)$ only for $s = a$, $t = b$ (no self-intersections). Thus a circle is simple, but a figure of eight is not.
3. A simple closed path γ is called a *contour.* Then $\int_\gamma f$ is called the *contour integral* of f round γ. From now on, we shall be dealing largely with contour integrals.

Continuity and Connectedness.
Example. In $\mathbf{R}$:
$f : \mathbf{R} \to \mathbf{R}$,
$f(x) := \begin{cases} 0 & (x < 0) \\ 1 & (x \geq 0) \end{cases}$.
(unit jump function, Heaviside function). f is continuous except at 0, where it has a jump discontinuity. Now modify this example by deleting the origin from the domain of definition:
$f : (-\infty, 0) \cup (0, \infty) \to \mathbf{R}$, $f(x) = \begin{cases} 0 & (x < 0) \\ 1 & (x > 0) \end{cases}$.

f is now *continuous*. Its only possible point of discontinuity is no longer there.

In $\mathbf{C}$:

$$f : D_1 := N\,(-1/2, 1/2) \cup \{0\} \cup N\,(1/2, 1/2) \to \mathbf{R},$$
$$f(z) = \begin{cases} 0 & (|z + \frac{1}{2}| < 1/2) \\ 1 & (z = 0 \text{ or } |z - \frac{1}{2}| < 1/2) \end{cases}$$

f is discontinuous at 0. This example is obviously badly behaved: f is discontinuous, and D_1 is not open.
But if

$$f : D_2 := N\,(-1/2, 1/2) \cup N\,(1/2, 1/2) \to \mathbf{R},$$
$$f \equiv 0 \text{ on } N(-1/2, 1/2), \quad f \equiv 1 \text{ on } N(1/2, 1/2).$$

f is now *continuous* (indeed, infinitely differentiable), and its domain is *disconnected*. This example is also very badly behaved – but for reasons that are *not* yet obvious. We will build a theory where, knowing the function values in *any* disc (however small) determines the function values anywhere (see the end of Lecture 1, where we advertised this as a complete contrast between Real Analysis and Complex Analysis). Such a theory must *exclude* examples such as the above. We do this by restricting the domain of the definition of a function to be *connected*. The above example then becomes not one function but *two* – one $f_1 \equiv 0$ on the left-hand disc, the other $f_2 \equiv 1$ on the other disc. Now everything is well-behaved, and we can build a good theory.

Note. The above complex examples do two things.

1. They motivate the definition of *domain*, below.
2. They illustrate that obvious bad behaviour is less dangerous than non-obvious bad behaviour – as it is easier to avoid. The first complex example is obviously bad. The second one looks innocuous at first sight – but is actually lethal.