m2pm3l16(11).tex
Lecture 16. 14.2.2011.

Defn. 1. A *domain* D is a non-empty, open, connected subset of $\mathbf{C}$ (domains are sometimes called *regions*.
These are the sets suitable as *domains of definition* for functions f differentiable in the sense of II.2.
2. A function f is *holomorphic* if it is differentiable (in the sense of II.2) in some domain D: $f : D \longrightarrow \mathbf{C}$. We then say f is *holomorphic in* D.
Note. D is non-empty (so a function can be defined on on it), open (differentiable $\leftrightarrow$ open domain: differentiability at a point requires us to form diference quotients at *all* nearby points; all must be in the domain of definition, which must thus be open) and connected (to exclude examples such as the above).
The Jordan Curve Theorem.

This states that a simple closed curve γ in $\mathbf{C}$ (simple: non-self-intersecting) divides the plane into two connected domains, one bounded (called the *inside* of γ, $I(\gamma)$), and one unbounded (called the *outside* of γ, $O(\gamma)$). This was stated by Camille JORDAN (1838-1922) in 1866, but only proved in 1905 by Oswald VEBLEN (1880-1960). The result is topological, and the subject of Topology only emerged in the early 1900s. A complete proof of the Jordan Curve Theorem is difficult. But if we quote it, our proof of Cauchy's Theorem for Triangles extends to general contours γ such that γ and its interior $I(\gamma)$ are in D.

The reason that triangles, star domains etc. are easier here is that one can handle their interiors $I(\gamma)$ from scratch, without the Jordan Curve Theorem. See the Handout on Cauchy's Theorem for more.

Lemma (ML Inequality). If $|f| \leq M$ on γ, and γ has length L, then

$$\left|\int_\gamma f\right| \leq ML.$$

Proof. As $|\int .| \leq \int |.|$,

$$\left|\int_\gamma f\right| = \left|\int_a^b f(\gamma(t))\dot{\gamma}(t)\,dt\right| \leq \int_a^b |f(\gamma(t))|.|\dot{\gamma}(t)|\,dt \leq M\int_a^b |\dot{\gamma}(t)|\,dt = ML. \quad //$$

Example (Fundamental Integral). For C on the unit circle (centre O),

$$\int_C z^n \, dz = \begin{cases} 2\pi i & (n = -1), \\ 0 & (n \neq -1). \end{cases}$$

Proof. Parametrize C by $z = e^{i\theta}$ $(0 \leq \theta \leq 2\pi)$, $dz = ie^{i\theta}\, d\theta$.

$$\int_C z^n \, dz = \int_0^{2\pi} e^{in\theta}.ie^{i\theta}\, d\theta = i\int_0^{2\pi} e^{i(n+1)\theta}\, d\theta = i\int_0^{2\pi} d\theta = 2\pi i \text{ if } n = -1 \quad (n{+}1 = 0).$$

But if $n \neq -1$, $n+1 \neq 0$, RHS $= i\left[e^{i(n+1)\theta}/(i(n+1))\right]_0^{2\pi} = 0$, by periodicity of $\cos\theta$, $\sin\theta$, $e^{i\theta}$.

5. Cauchy's Theorem.

Theorem (Cauchy's Theorem for Triangles). If f is holomorphic in a domain D containing a triangle γ and its interior $I(\gamma)$ – then

$$\int_\gamma f = 0.$$

Proof. Join the three midpoints of the sides of the triangle γ. This *quadrisects* γ into 4 *similar* triangles. Call these γ_1 to γ_4: then

$$\int_\gamma f = \sum_1^4 \int_{\gamma_i} f.$$

For, RHS contains 12 terms, 3 for each of the 4 triangles. The 'outer 6' add to $\int_\gamma f$; the 'inner 6' cancel in pairs. So, for at least one i,

$$\left|\int_{\gamma_i} f\right| \geq \frac{1}{4}|I|, \text{ where } I = \int_\gamma f.$$

For if not, each $\left|\int_{\gamma_i} f\right| < \frac{1}{4}|I|$, so

$$|I| = \left|\int_\gamma f\right| = \left|\sum_1^4 \int_{\gamma_i} f\right| \leq \sum_1^4 |\int_{\gamma_i} f| < \sum_1^4 \frac{1}{4}|I| = |I|,$$

a contradiction. W.l.o.g., take this i as 1. So: $\left|\int_{\gamma_1} f\right| \geq \frac{1}{4}|I|$.
Now quadrisect γ_1. Repeating the argument above, at least one of the 4 resulting triangles, γ_2 say, has

$$\left|\int_{\gamma_2} f\right| \geq \frac{1}{4}\left|\int_{\gamma_1} f\right| \geq \frac{1}{4^2}|I|.$$