m2pm3l17(11).tex
Lecture 17. 15.2.2011.

Continue (or use induction): we obtain a sequence of triangles $\gamma_1, \gamma_2, ..., \gamma_n, ...$ s.t. if Δ denotes the union of γ and its interior $I(\gamma)$ and similarly for Δ_n, γ_n,

$$\Delta_{n+1} \subset \Delta_n \subset ... \subset \Delta_2 \subset \Delta_1 \subset \Delta;$$

lengths: $L(\gamma_n) = 2^{-n}L$ ($L = L(\gamma)$, length of γ); and $4^{-n}|I| \leq \left|\int_{\gamma_n} f\right|$.
The sets Δ_n are decreasing, closed and bounded (so *compact*), and non-empty. So by *Cantor's Theorem* (Handout, or I.2.6), $\bigcap_{n=1}^{\infty} \triangle_n \neq \emptyset$.
Take $z_0 \in \bigcap_{n=1}^{\infty} \triangle_n$. As Δ is compact, $z_0 \in \Delta$. Since *by assumption* f is holomorphic in D, f is holomorphic at z_0. So

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ s.t. } \forall z \text{ with } 0 < |z - z_0| < \delta, \left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \epsilon :$$

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| < \epsilon |z - z_0|.$$

As $diam(\gamma_n) = 2^{-n} \downarrow 0$ as $n \to \infty$, Δ_n tends to $\{z_0\}$ as $n \to \infty$. So $\Delta_N \subset N(z_0, \delta)$ for all large enough n. For this n, and all $z \in \Delta_n$, $|z - z_0| \leq L(\gamma_n) = 2^{-n}L$ (Triangle Lemma, Problems 4). Now

$$f(z_0) + f'(z_0)(z - z_0) = F'(z), \text{ with } F(z) = f(z_0)(z - z_0) + \frac{1}{2}f'(z_0)(z - z_0)^2.$$

$$\begin{aligned}
\int_{\gamma_n} [f(z_0) + f'(z_0)(z - z_0)]\, dz &= \int_{\gamma_n} F'(z)\, dz \\
&= \int_a^b F'(z(t))\dot{z}(t)\, dt && \text{(if } \gamma_n \text{ is parametrised by } [a, b]) \\
&= [F(z(t))]_{t=a}^b && \text{(Fundamental Th. of Calculus)} \\
&= F(z(b)) - F(z(a)) \\
&= F(z(a)) - F(z(a)) && (z(b) = z(a) \text{ as triangle } \gamma_n \text{ is closed}) \\
&= 0.
\end{aligned}$$

This and and $(*)$ give

$$\left|\int_{\gamma_n} f\right| < \epsilon \int_{\gamma_n} |z - z_0|\, dz.$$

But

$$\begin{aligned}
\int_{\gamma_n} |z - z_0|\, dz &\leq \max_{\gamma_n} |z - z_0| \cdot L(\gamma_n) && \text{(ML)} \\
&\leq L(\gamma_n) \cdot L(\gamma_n) && \text{(Triangle Lemma, Problem Sheet)} \\
&\leq 4^{-n}L^2 && (L(\gamma_n) \leq L \cdot 2^{-n}).
\end{aligned}$$

Combining:

$$4^{-n}|I| \leq \left|\int_{\gamma_n} f\right| \leq \epsilon \cdot 4^{-n} \cdot L^2. \qquad \text{(ii)}$$

By (i) and (ii):

$$4^{-n}|I| \leq \epsilon 4^{-n} L^2 : \qquad |I| \leq \epsilon \cdot L^2.$$

But $\epsilon > 0$ is arbitrarily small. So $|I| = 0$: $I = 0$: $\int_\gamma f = 0$. //

Cor. (Cauchy's Theorem for Rectangles). If f is holomorphic in a domain D containing a rectangle R and its interior - then $\int_R f = 0$.

Proof. Bisect into two triangles, γ_1, γ_2: $\int_R f = \int_{\gamma_1} f + \int_{\gamma_2} f = 0 + 0 = 0$. //

Similarly one obtains *Cauchy's Therem for Polygons*: Triangulate.
Defn. 1. If $z_1, z_2 \in \mathbf{C}$, write $[z_1, z_2]$ for the *line segment* joining them in $\mathbf{C}$.
2. A domain D is *star-shaped* (or, is a *star domain*) with *star-centre* $z_0 \in D$ if, for all $z \in D$, the line-segment $[z_0, z] \subset D$.
E.g. Discs are star-shaped. Convex sets are star-shaped.
E.g: $D =$ 'Union of two athletics tracks' – star shaped with star-centre z_0, but not convex.

Theorem (of the Antiderivative). If f is holomorphic in a star-domain D with star-centre z_0,

$$F(z) = \int_{[z_0,z]} f,$$

then $F' = f$: F is an antiderivative of f, and f is the derivative of F.

Proof. Take any $z_1 \in D$. We prove $F'(z_1)$ exists and is $f(z_1)$. As D is open and $z_1 \in D$, some neighbourhood $N(z_1, \epsilon_1) \subset D$. For $|h| < \epsilon_1$, $z_1 + h \in D$. As $z_0, z_1, z_1 + d \in D$, the line-segments $[z_0, z_1]$, $[z_0, z_1 + h] \subset D$ (D is star-shaped with star-centre z_0).

Let γ be the triangle with vertices z_0, z_1, z_1+h, Δ be the union of γ and its interior $I(\gamma)$. Then $\Delta \subset D$. By Cauchy's Theorem for triangle Δ, $\int_\gamma f = 0$. That is,

$$\int_{[z_0,z_1]} f + \int_{[z_1,z_1+h]} f + \int_{[z_1,z_1+h,z_0]} f = 0 : \quad F(z_1) + \int_{[z_1,z_1+h]} f - F(z_1+h) = 0.$$

So

$$\frac{F(z_1 + h) - F(z_1)}{h} = \frac{1}{h} \int_{[z_1,z_1+h]} f. \qquad (i)$$