m2pm3l19(11).tex
Lecture 19. 21.2.2011.

Note. In the argument above, γ was not a contour, as going along P both ways introduces a self-intersection. We can do either of:
(i) Appeal instead to a different form of Cauchy's Theorem (involving homology - see Handout);
(ii) Change to an 'approximating contour'. This new contour has no points of self-intersection (it *is* a contour). By continuity the change in the contour integral can be made arbitrarily small.

6. Cauchy's Integral Formulae.

Theorem (Cauchy's Integral Formula, CIF). If f is holomorphic inside and on a positively oriented contour γ, and a is inside γ ($a \in I(\gamma)$), then:

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz.$$

Proof. As a is inside γ, and $I(\gamma)$ is open, $N(a,R) \subset I(\gamma)$ for some $R > 0$. For any r, $0 < r < R$, and $\gamma(a,r)$ the circle centre a radius r,

$$\int_\gamma \frac{f(z)}{z-a}\, dz = \int_{\gamma(a,r)} \frac{f(z)}{z-a}\, dz \qquad \text{(Deformation Lemma).} \qquad (i)$$

Using the Deformation Lemma again, and parametrising the circle $\gamma(a,r)$ by $z = a + re^{i\theta}$, $0 \le \theta \le 2\pi$, so $dz = ire^{i\theta}\, d\theta$,

$$\int_\gamma \frac{f(a)}{z-a}\, dz = \int_{\gamma(a,r)} \frac{f(a)}{z-a}\, dz = f(a) \int_{\gamma(a,r)} \frac{1}{z-a}\, dz = f(a) \int_0^{2\pi} i\, d\theta$$
$$= 2\pi i f(a). \qquad (ii)$$

So by (i) and (ii),

$$\left| \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz - f(a) \right| = \left| \frac{1}{2\pi i} \int_{\gamma(a,r)} \frac{f(z)-f(a)}{z-a} dz \right| = \left| \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a+re^{i\theta}) - f(a)}{re^{i\theta}} ire^{i\theta}\, d\theta \right|,$$

which by ML is

$$\le \frac{1}{2\pi} \cdot 2\pi \sup_{[0,2\pi]} |f(a+re^{i\theta}) - f(a)| \to 0 \qquad (r \to 0),$$

by uniformly continuity of f on $\gamma \cup I(\gamma)$. So $LHS = 0$. //

Corollary. The values of f *on* the contour determine the values of f *everywhere inside* the contour.

This has further important consequences. The next result is Liouville's Theorem (Joseph LIOUVILLE (1809-1882), lectures in 1847 – actually published by Cauchy in 1844).

Theorem (Liouville's Theorem). If f is holomorphic throughout $\mathbf{C}$, and bounded – then f is *constant*.

Proof. Take M with $|f(z)| \le M\ \forall z \in \mathbf{C}$. Take any $z_1, z_2 \in \mathbf{C}$. Choose $R \ge 2\max(|z_1|, |z_2|)$. Then for $|z| = R$, $|z - z_1| \ge \frac{1}{2}R$, $|z - z_2| \ge \frac{1}{2}R$ by the Triangle Inequality. By CIF with $\gamma = \gamma(0, R)$,

$$f(z_1) - f(z_2) = \frac{1}{2\pi i}\int_\gamma f(z)\left(\frac{1}{z - z_1} - \frac{1}{z - z_2}\right) dz = \frac{z_1 - z_2}{2\pi i}\int_\gamma \frac{f(z)}{(z - z_1)(z - z_2)}\, dz.$$

So by ML,

$$|f(z_1) - f(z_2)| \le \frac{|z_1 - z_2|}{2\pi} \cdot 2\pi R \cdot M \cdot \frac{2}{R} \cdot \frac{2}{R} = \frac{4M|z_1 - z_2|}{R} \to 0 \qquad (R \to \infty).$$

So LHS$= 0$: $f(z_1) = f(z_2)$: f constant. //

Note. Constants are trivial. Liouville's Theorem says that a non-trivial function, holomorphic everywhere, is *unbounded*.

Theorem (Fundamental Theorem of Algebra). A (complex) polynomial $p_n(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$ $(a_i \in \mathbf{C},\ a_n \neq 0)$ of degree n *factorizes*, as

$$p_n(z) = a_n \prod_{k=1}^{n} (z - z_k) \quad (z_k \in \mathbf{C}).$$

That is, a polynomial of degree n has (exactly) n roots, possibly complex, and counted according to multiplicity.

Proof. Assume $p_n(z)$ has *no* roots. Then $p_n(z) \neq 0, \forall z$. So $1/p_n(z)$ is defined, for all z, and holomorphic, for all z.