m2pm3l20(11).tex
Lecture 20. 22.2.2011.

Proof of the Fundamental Theorem of Algebra (continued).

As $|z| \to \infty$, $|p_n(z)| \sim |a_n z^n| = |a_n||z|^n \to \infty$. So $|1/p_n(z)| \to 0$ ($|z| \to \infty|$), hence it is ≤ 1, for $|z|$ large enough: $|z| \geq C$, say. (i)
But $1/p_n(z)$ is (holomorphic, so) continuous for $|z| \leq C$, so as $\{z : |z| \leq C\}$ is closed and bounded, it is *compact* (Heine-Borel), so $1/p_n(z)$ is *bounded* on $|z| \leq C$. (ii)
By (i) and (ii): $1/p_n(z)$ is *bounded throughout* $\mathbf{C}$. As $1/p_n(z)$ is *holomorphic*, $1/p_n(z)$ is *constant*, by Liouvilles' Theorem. So $p_n(z)$ is constant.
But polynomials (of positive degree) are not constant. Contradiction.
So $p_n(z)$ has at least one root, z_n say:

$$p_n(z) = (z - z_n)p_{n-1}(z),$$

for some polynomial p_{n-1} of degree $n-1$. Continuing in this way, or by induction, p_n factorises:

$$p_n(z) = a_n(z - z_n)(z - z_{n-1})...(z - z_1). \quad //$$

Defn. If $f(z)$ is holomorphic throughout $\mathbf{C}$, f is called *entire* (=integral).
So Liouville's Theorem says: entire + bounded $\Rightarrow$ constant.

Theorem (Cauchy's Integral Formula for the First Derivative), CIF(1).
Let f be holomorphic inside and on a positively oriented contour γ. Then f' is holomorphic inside γ, and

$$f'(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^2}\,dz \qquad (a \text{ inside } \gamma).$$

Proof. By CIF and its Proof, for $r > 0$ small enough,

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)}\,dz = \frac{1}{2\pi i}\int_{\gamma(a,r)} \frac{f(z)}{(z-a)}\,dz,$$

and similarly for $f(a+h$ with $|h|$ small enough ($|h| < r$). So

$$\frac{f(a+h)-f(a)}{h} = \frac{1}{2\pi i h}\int_{\gamma(a,r)} f(z)\left[\frac{1}{z-a-h} - \frac{1}{z-a}\right] dz$$

$$= \frac{1}{2\pi i}\int_{\gamma(a,r)} \frac{f(z)}{(z-a-h)(z-a)}\,dz \to \frac{1}{2\pi i}\int_{\gamma(a,r)} \frac{f(z)}{(z-a)^2}\,dz \qquad (h \to 0)$$
$$= \frac{1}{2\pi i}\int_{\gamma} \frac{f(z)}{(z-a)^2}\,dz$$

(Deformation Lemma - or as in Proof of CIF). //

Theorem. If f is holomorphic in a domain D, then:
(i) $f^{'}$ is holomorphic in D,
(ii) f is *infinitely differentiable* in D: $f^{'}$, $f^{''}$,...,$f^{(n)}$,... are holomorphic in D for all n.

Proof. (i) Choose $a \in D$. Then choose a positively oriented contour γ containing a and lying in D. Then use CIF.
(ii) By (i) for $f^{'}$, $f^{''}$ is holomorphic in D. Continue in this way, or by induction. //

Compare Real Analysis! There, $C, C^1, ..., C^n$ (n continuous derivatives), ..., C^∞ are *all different*. Here they are (essentially) *all the same*.

Theorem (Cauchy's Integral Formula for the nth Derivative, CIF(n)). In CIF(1),

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_{\gamma} \frac{f(z)}{(z-a)^{n+1}}\,dz \qquad (n = 0, 1, 2, ...).$$

Proof. By induction, or as in the Proof of CIF(1). //

There is partial converse to Cauchy's Theorem (Giacinto MORERA (1856-1909), in 1889).

Theorem (Morera's Theorem). If f is continuous in a domain D, and $\int_\gamma f = 0$ for all triangles γ in D – then f is holomorphic in D.

Proof. Take $a \in D$, $r > 0$ s.t. $N(a,r) \subset D$; and for $z \in N(a,r)$, $F(z) = \int_{[a,z]} f$ (defined, as f is continuous). Integrating round the triangle (as in the proof of the Theorem of the Antiderivative): F is an antiderivative of f: F is holomorphic, with $F^{'} = f$. By CIF(1)(i) applied to F, $F^{'} = f$ is holomorphic. //