m2pm3l21.tex
Lecture 21. 24.2.2011.

7. Cauchy-Taylor Theorem.

Theorem (Cauchy's Form of Taylor's Theorem: Cauchy-Taylor Thm.) If f is holomorphic in $N(a, R)$ $(R > 0)$, there exists constants c_n s.t.

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \qquad (z \in N(a, R)),$$

where the coefficients c_n are given by

$$c_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}}\, dz \qquad (\gamma = \gamma(a, r),\ 0 < r < R).$$

Proof. Choose $z \in N(a, R)$, and then r with $|z - a| < r < R$. By CIF,

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z}\, dz.$$

But

$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{(w-a)} \frac{1}{1 - \frac{z-a}{w-a}} = \frac{1}{w-a} \cdot \sum_0^\infty \frac{(z-a)^n}{(w-a)^n}.$$

On $\gamma = \gamma(a, r)$, $|w - a| = r > |z - a|$. So the series converges *uniformly* on w. So we can interchange $\int_\gamma \ldots dw$ and $\sum_0^\infty$:

$$f(z) = \frac{1}{2\pi i} \int_\gamma f(w) \cdot \sum_0^\infty \frac{(z-a)^n}{(w-a)^{n+1}}\, dw = \sum_{n=0}^{\infty} (z-a)^n \cdot \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^{n+1}}\, dw.$$

So (i) $f(z) = \sum_0^\infty c_n (z-a)^n$, where (ii) $c_n = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^{n+1}}\, dw = \frac{f^{(n)}(a)}{n!}$, by CIF($n$). //

We already know [I.2.11] that a power series can be differentiated infinitely often. So the following are equivalent:
(i) f is holomorphic (differentiable once) throughout D;
(ii) f is differentiable infinitely often throughout D;
(iii) f is the sum its Taylor series throughout D ('f is *analytic*' in D).

Compare the real case (Lecture 1, $f(x) = \exp(-1/x^2)$)!
Note. Property (iii) – f is *analytic* – is equivalent to (i) – f is holomorphic. So terms analytic and holomorphic can be used interchangeably.
Defn. If $f(z) = \sum_0^\infty c_n(z-a)^n$ and $c_0 = c_1 = ... = c_{k-1} = 0$, but $c_k \neq 0$, i.e. $f(z) = c_k(z-a)^k[1 + c_{k+1}(z-a) + ...]$, then we say that f *has a zero of order* k *at* a. Then $f(z) = (z-a)^k g(z)$, g holomorphic, $g(a) \neq 0$.

Theorem. If f is holomorphic and not $\equiv 0$, then the zeros of f are *isolated*: each zero a has a neighbourhood containing no zeros other then itself.

Proof. If a is a zero of f of order k, $f(z) = (z-a)^k g(z)$, g holomorphic, $g(a) \neq 0$. As g is continuous, $g(\cdot) \neq 0$ in some neighbourhood of a. So $f(z) = (z-a)^k g(z) \neq 0$ in this neighbourhood except at a. //

Cor. 1. If f is holomorphic in D, and has zeros z_n with a limit point $z_0 \in D$ – then $f \equiv 0$.

Proof. By the Cauchy-Taylor Theorem, if $f^{(n)}(a) = 0$ for all $n = 0, 1, 2, \ldots$, then $f \equiv 0$ in some neighbourhood of a. So the set U of points at which f and all its derivatives vanish is *open* (each point a of U has a neighbourhood of points of U). If the complement of U in D is $V := D \setminus U$, V is the set of points at which some derivative of f is non-zero. Each such derivative is continuous (indeed, it is holomorphic). So if $a \in V$ with $f^{(n)}(a) \neq 0$, by continuity $f^{(n)}$ is also non-zero at all points close enough to a. This says that V is open. So the set D, connected by definition of domain, is the disjoint union of two open sets, U and V. By connectedness (II.3 , Lecture 14), one of them must be empty. By the Theorem, U is non-empty (the zeros of f are not isolated, so f must be $\equiv 0$ in some neighbourhood). So V is empty. So $f \equiv 0$. //

Cor. 2 (Identity Theorem). If f_1, f_2 are holomorphic, and $f_1(z_n) = f_2(z_n)$ at z_n with a limit point $z_0 \in D$ – then $f_1 \equiv f_2$ in D.

Proof. Apply Corollary (1) to $f_1 - f_2$. //

Cor. 3. A holomorphic function is uniquely determined by its values in *any arbitrarily small disc.* Indeed, any infinite set with a limit point in the domain of holomorphic will do.

Note. Recall the example in Section 2.3 Connectedness. The above results *only work* because we have restricted the domains D to be *connected.*
Harmonic functions and holomorphic functions. Call $u(x, y)$ *harmonic* in D, $u \in \mathcal{H}(D)$, if it has continuous 2nd-order partials, and satisfies Laplace's equations: $\Delta u := u_{xx} + u_{yy} = 0$. As in II.2, given u, we can find f holomorphic ($f \in \mathcal{H}(D)$) and $v \in \mathcal{H}(D)$ s.t. $f = u + iv$, $u, v \in \mathcal{H}(D)$.

Recall that in II.2, we saw that continuity of partials of u, v (and the CR equations) was equivalent to holomorphy of f. We now know that this is equivalent to f being infinitely differentiable, and so to u, v being infinitely differentiable. So we now know that assuming continuity of 2nd-order partials (so as to have $u_{xy} = u_{yx}$, used in the proof that the CR equations imply u, v harmonic) is in fact no restriction.