m2pm3l22(11).tex
Lecture 22. 28.2.2011.

8. Analytic Continuation.
Recall (II.7, Lecture 21):
(i) Given f holomorphic in D, to expand f about $z_0 \in D$: the resulting expansion has R of C R, the distance from z_0 to the nearest singularity of f.
(ii) Given f_1, f_2, equal on *infinitely many* points z_n with limit $z_0 \in D$ – then $f_1 \equiv f_2$ in D.

Note that $z_0 \in D$ is essential here. Eg, $\sin \pi z = 0$ for $z = n \in \mathbf{Z}$. So $\sin(1/(\pi z))$ is holomorphic in $D := \mathbf{C} \backslash \{0\}$, and $= 0$ for $z = 1/(n\pi) \to 0 \notin D$, the domain of holomorphy of the function. But $\sin(1/(\pi z))$ is *not* $\equiv 0$!

Suppose now:
(i) f_1 is holomorphic on a domain D_1,
(ii) f_2 is holomorphic on a larger domain D_2 with $D_1 \subset D_2$,
(iii) $f_1 = f_2$ on D_1.
Then we may *extend* the domain of definition of f_1 from D_1 to D_2, by taking $f_1(z) := f_2(z)$ in $D_2 \setminus D_1$. By the Identity Theorem, no ambiguity can be introduced. So we lose nothing, and *gain* something, by extending the domain of f_1. This process, due to Weierstrass, is called *analytic continuation.*
1. *Analytic continuation by power series.*

To illustrate this process, we take the simplest possible power series – the *geometric series*: $\sum_{n=0}^{\infty} z^n = 1/(1-z)$. The power series on the left-hand side has R of C 1, so is defined for $\{z : |z| < 1\}$, the unit disc. The RHS is defined for $\mathbf{C} \setminus \{1\}$ – a much bigger domain.

Take a in the unit disc, and power-series expand the function given by $\sum_0^\infty z^n$ about $z = a$:

$$\frac{1}{1-z} = \frac{1}{(1-a)-(z-a)} = \frac{1}{1-a} \cdot \frac{1}{1-\frac{z-a}{1-a}} = \sum_0^\infty \frac{(z-a)^n}{(1-a)^{n+1}}.$$

This power series converges in the disc centre a touching the unit circle – which has radius $1-|a|$. But it actually converges in the larger disc, centre a and radius $|1-a|$, the distance from the new base-point a to the singularity at 1 (the RHS converges for $\{z : |z-a| < |1-a|\}$, so the R of C is $|1-a|$).
1. Taking a near -1 (from the right), we can cover the disc $N(-1, 2)$ (on $\mathbf{R}$, goes from -3 to $+1$). Taking a near -3 (from the right), we can cover the disc $N(-3, 4)$ (on $\mathbf{R}$, goes from -7 to $+1$). And so on: continuing this way,

we expand the domain of definition to the *half-plane* $x = Re\ z < 1$.
2. Now expand about points $z_n = \pm i\ n + (1 - 1/n)$. The R of C is the distance from z_n to 1. The union of these discs of convergence is $\mathbf{C} \setminus (1, \infty)$.
3. Now expand near the x-axis $(1, \infty)$. Again, R of C = distance from this point to 1. The union of these discs of convergence is $\mathbf{C} \setminus \{1\}$.
Check: $1/(1-z)$ is holomorphic on $\mathbf{C} \setminus \{1\}$!

We *identify* the function $1/(1-z)$, holomorphic on $\mathbf{C} \setminus \{1\}$, with *all these power series expansions*. Similarly for the general case:

A homomorphic function **is** *the set of all its power-series expansions.*

This is the Weierstrass approach to analytic functions – via power series. We do not need it for this example, which we can 'sum on sight'; we do need it in general, where we cannot.
2. *Analytic continuation by integrals.*
a. *The Gamma function.*

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt \qquad (x > 0), \qquad \Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt \qquad (Re\ z > 0).$$

Now $\Gamma(z+1) = z\Gamma(z)$ (check – integration by parts). So $\Gamma(z) = \Gamma(z+1)/z$. RHS is defined for $Re\ z > -1$ (with a singularity at 0, from $1/z$). So we can *define* $\Gamma(z)$ for $Re\ z > -1$ by LHS. Then RHS is defined for $Re\ z > -2$ (with singularities at 0 from $1/z$, -1 from $\Gamma(z)$). So we can *define* $\Gamma(z)$ by LHS for $Re\ z > -2$. Continuing in this way: we can *define* $\Gamma(z)$ throughout $\mathbf{C}$ (with singularities at $z = 0, -1, -2, \ldots, -n, \ldots$). This gives the *analytic continuation* of $\Gamma(z)$ to the whole plane.
b. *The Logarithm.*

$$\log z := \int_{[1,z]} \frac{1}{w}\, dw \qquad (\log 1 = 0,\ e^0 = 1).$$

This definition succeeds wherever we can join z to 1 by a straight-line segment which does not go through the inevitable logarithmic singularity at $z = 0$ – that is, *except* for z on the negative real axis or 0, i.e. for $z \in \mathbf{C} \setminus (-\infty, 0]$, the *cut plane* $\mathbf{C}_{cut}$: $\log z$ is holomorphic in $\mathbf{C}_{cut}$. This is best possible: $\log(re^{i\theta}) = \log r + i\theta$ is *discontinuous* across the cut, so is far from being holomorphic on the cut. For just above the cut, $\theta = \arg z$ approaches π; just below the cut, it approaches $-\pi$; there is a discontinuity of $2\pi i$ in the logarithm across the cut. The cut is a limit to analytic continuation – a *natural boundary* (Exam 2009, Q2).