m2pm3l23(11).tex

Lecture 23. 1.3.2011.

3. *Analytic continuation by Dirichlet series.*

A *Dirichlet series* is one of the form $\sum_{n=1}^{\infty} a_n/n^s$, with a_n and s complex; we always write $s = \sigma + i\tau$. We quote that in the complex s-plane:
(i) the domain of absolute convergence is a half-plane of the form $Re\ s > \sigma_a$;
(ii) the domain of (conditional) convergence is a half-plane $Re\ s = \sigma_c$ (where $-\infty \le \sigma_c \le \sigma_a \le +\infty$);
(iii) the Dirichlet series is holomorphic in its half-plane of convergence.
These results (for which see any book on Analytic Number Theory) depend on the Abel and Dirichlet tests of convergence of series (see Handout on website).

Example: The Riemann zeta function: $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$.

For $s = \sigma$ real, this is convergent iff $\sigma > 1$, so $\sigma_a = 1$ and the half-plane of absolute convergence is $\sigma > 1$. Now for $s = \sigma$ real,

$$\sum_{n=1}^{\infty} (-)^{n-1}/n^{\sigma} = \sum_{odd} 1/n^{\sigma} - \sum_{even} 1/n^{\sigma} = \sum_o - \sum_e,$$

say. Now $\sum_e = \sum_1^{\infty} 1/(2n)^{\sigma} = 2^{-\sigma} \sum_1^{\infty} 1/n^{\sigma} = 2^{-\sigma}\zeta(\sigma)$. So

$$\sum_{n=1}^{\infty} (-)^{n-1}/n^{\sigma} = \sum_o - \sum_e = \sum_o - 2^{-\sigma}\zeta(\sigma), \qquad \zeta(\sigma) = \sum_o + \sum_e = \sum_o + 2^{-\sigma}\zeta(\sigma).$$

Subtract:

$$\sum_{n=1}^{\infty} (-)^{n-1}/n^{\sigma} - \zeta(\sigma) = -2.2^{-\sigma}\zeta(\sigma) = -2^{1-\sigma}\zeta(\sigma):$$

$$\sum_{n=1}^{\infty} (-)^{n-1}/n^{\sigma} = (1 - 2^{1-\sigma})\zeta(\sigma):$$

$$\zeta(\sigma) = (1 - 2^{1-\sigma})^{-1}.\sum_{n=1}^{\infty} (-)^{n-1}/n^{\sigma}. \qquad (*)$$

Note that the series on RHS converges in $\sigma > 0$, by the Alternating Series Test. Now let s be complex, and define

$$\zeta(s) := (1 - 2^{1-s})^{-1}.\sum_{n=1}^{\infty} (-)^{n-1}/n^{s}.$$

By (iii) above, the second factor on RHS is holomorphic in $\sigma > 0$ (the first factor is holomorphic as $2^{-s} = e^{-s \log 2}$ is). So we may use $(*)$ to *continue* $\zeta(s)$ *analytically* from $\sigma > 1$ to $\sigma > 0$.

Note. 1. It turns out that the behaviour of ζ in the strip $0 < \sigma < 1$, and in particular on and near the *critical line* $\sigma = 1/2$, is crucially important in the study of the distribution of prime numbers – e.g., the Prime Number Theorem (PNT) in Analytic Number Theory.
2. We quote (e.g., from the functional equation for the zeta function): $\zeta(z)$ can be continued analytically to the whole plane, where it is holomorphic except for a singularity at $z = 1$ (a simple pole of residue 1 – see §10 below).

4. *Analytic continuation by identities.*
Example: The Gamma function.
Recall $\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}\,dt$. By Real Analysis (Problems 4, Q4),

$$\Gamma(x)\Gamma(1-x) = \int_0^\infty \frac{v^{x-1}}{1+v}\,dv \qquad (0 < x < 1).$$

By Complex Analysis (III.6 below),

$$\int_0^\infty \frac{v^{x-1}}{1+v}\,dv = \frac{\pi}{\sin \pi x} \qquad (0 < x < 1).$$

Combining:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} \qquad (0 < x < 1).$$

As $(0,1) \subset \mathbf{C}$ is infinite and contains limit points (all its points are limit points!) we can continue analytically:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \qquad (z \in \mathbf{C}).$$

(*Euler's reflection formula for the Gamma function* – Exam 2010, Q3).
So RHS has no zeros (or sin would have a pole). So LHS has no zeros:

Cor. $\Gamma(z)$ has no zeros for $z \in \mathbf{C}$ and has poles at $z = 0, -1, -2, \ldots$.

So $1/\Gamma(z)$ has zeros at $z = 0, -1, -2\ldots$ and no poles. Hence $1/\Gamma(z)$ is entire, and

$$\frac{1}{\Gamma(z)}.\frac{1}{\Gamma(1-z)} = \frac{\sin \pi z}{\pi}.$$

Here $1/\Gamma(z)$ is entire with zeros at $0, -1, -2, \ldots$, $1/\Gamma(1-z)$ is entire with zeros at $1, 2, \ldots$, and $\sin \pi z/\pi$ is entire with zeros at the integers. Then $\Gamma(z)$, $\Gamma(1-z)$, $\sin \pi z/\pi$ are meromorphic with the corresponding zeros, but no poles.