m2pm3l24(11).tex

Lecture 24. 3.3.2011.

9. The Maximum Modulus Theorem. See Website - not examinable.

10. Laurent's Theorem and Singularities.

There may be points where f is *not* holomorphic. There, a new kind of expansion is needed (Pierre-Alphonse LAURENT (1813-54), in 1843).

Theorem (Laurent's Theorem). If f is holomorphic in a domain D containing the annulus $0 < R' \leq |z-a| \leq R < \infty$ – then f has an expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n \qquad (R' \leq |z-a| \leq R),$$

where

$$c_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}} \, dz \qquad (n \in \mathbf{Z})$$

and γ is a positively oriented contour in the annulus surrounding a.

Proof. Write C, C' for the circles centre a radius R, R', γ for the closed path consisting of: (i) C anticlockwise (+ve sense); (ii) a line segment L from C to C'; (iii) C' clockwise (-ve sense); (iv) L reversed, to get back to the starting point on C. The open annulus is the interior $I(\gamma)$ of γ. By CIF,

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z} \, dw = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z} \, dw - \frac{1}{2\pi i} \int_{C'} \frac{f(w)}{w-z} \, dw$$

(the terms on RHS are from (i) and (iii), since those from (ii) and (iv) cancel).

On C, $|w-a| = R > |z-a|$, so $1/(w-z) = \sum_0^\infty (z-a)^n/(w-a)^{n+1}$ (as in the Proof Cauchy-Taylor Theorem). So

$$\begin{aligned}
\int_C \frac{f(w)}{w-z} \, dw &= \int_C f(w) \sum_0^\infty \frac{(z-a)^n}{(w-a)^{n+1}} \, dw \\
&= \sum_0^\infty \frac{(z-a)^n}{2\pi i} \int_C \frac{f(w)}{(w-a)^{n+1}} \, dw \qquad \text{(by uniform convergence)} \\
&= \sum_0^\infty c_n (z-a)^n, \quad c_n = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-a)^{n+1}} \, dw.
\end{aligned}$$

Similarly, on C', $|w-a| = R' < |z-a|$, $1/(z-w) = \sum_0^\infty (w-a)^n/(z-a)^{n+1}$:

$$-\frac{1}{2\pi i}\int_{C'} \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_{C'} \frac{f(w)}{z-w}\,dw = \frac{1}{2\pi i}\int_{C'} f(w)\cdot \sum_{m=0}^{\infty} \frac{(w-a)^m}{(z-a)^{m+1}}\,dw.$$

Interchanging $\int$ and $\sum$ by uniform convergence as before, this gives

$$\sum_{m=0}^{\infty} (z-a)^{-m-1}\cdot \frac{1}{2\pi i}\int_{C'} f(w)(w-a)^m\,dw = \sum_{n=-\infty}^{-1} (z-a)^n \cdot \frac{1}{2\pi i}\int_{C'} \frac{f(w)}{(w-a)^{n+1}}$$

(writing $n := -m-1$). Combining,

$$f(z) = \sum_{n=\infty}^{\infty} c_n (z-a)^n, \qquad c_n = \frac{1}{2\pi i}\int_{C \text{ or } C'} \frac{f(w)}{(w-a)^{n+1}}.$$

Here, each $\int_{C \text{ or } C'} = \int_\gamma$ (Deformation Lemma). //

Examples. 1. $f(z) = \exp(-1/z^2)$, in any annulus $0 < R' \le |z| \le R \le \infty$: a *Laurent series* of f at 0. Recall (Lecture 1) f is *very badly behaved* at 0.
2. *Bessel functions* (Exam 2010 Q2; Problems 7).

$$\exp\Big(\frac{1}{2}z(t-\frac{1}{t})\Big) = \sum_{n=-\infty}^{\infty} t^n J_n(z).$$

The LHS gives the *generating function* (GF) of the Bessel functions $J_n(z)$.
Defn. In the Laurent series $\sum_{-\infty}^{\infty} c_n(z-a)^n$ above: $\sum_0^\infty c_n(z-a)^n$ (holomorphic, by Cauchy-Taylor) is called the *analytic*, or *regular*, or *holomorphic* part of f at a; $\sum_{-\infty}^{-1} c_n(z-a)^n$ is called the *singular* part.

If the singular part is a *polynomial* in $1/(z-a)$ of degree n, we say f has a *pole* at a of *order* n. $m=1$: simple pole, $m=2$: double pole, etc.
When the singular part is not a polynomial – i.e., $c_n \ne 0$ for *infinitely many* negative n – then a is called an *essential singularity.* Essential singularities are points of *very bad behaviour.* (Branch-points are even worse!)
Examples. 1. $\sin z/z$ at 0: $\sin z/z \to 1$ $(z \to 0)$.
In such situations, if the function is *either* not defined at the point, *or* defined with a value making the function discontinuous there, then the point is a singularity. We then *remove* the singularity by defining the function to have the right value – a *removable singularity.*
2. $\sin(1/z)$ has *zeros* at $1/z = n\pi$, $z = 1/(n\pi)$; $1/\sin(1/z)$ has *poles* at $z = 1/n\pi \to 0$. Limits of poles are also points of very bad behaviour, and are also called *essential singularities.*