m2pm3l25(11).tex
Lecture 25. 7.3.2011.

11. Cauchy's Residue Theorem (CRT).

Defn. If f has a singularity at a, with singular part $\sum_{-\infty}^{-1} c_n(z-a)^n$, the coefficient c_{-1} of $1/(z-a)$ is called the *residue* of f at a, $Res_a f$.

Theorem (Cauchy's Residue Theorem, CRT). If f is holomorphic in a domain D except for finitely many singularities z_i, and γ is a contour with $\gamma \cup I(\gamma) \subset D$ – then

$$\int_\gamma f = 2\pi i \sum Res f,$$

where the sum is over the singularities z_i *inside* γ.

Proof. Surround each singularity z_i inside γ by a small circle γ_i, centre z_i, radius r_i. Let Γ be the closed path consisting of: (i) γ, anticlockwise (+ve sense); (ii) a path L_i from γ to γ_i; (iii) γ_i clockwise (-ve sense); (iv) L_i reversed back to γ (one γ_i, L_i for each singularity inside γ).

As in the proof of Laurent's Theorem, $\int_\Gamma f = 0$ (Cauchy's Theorem). But

$$0 = \int_\Gamma f = \int_\gamma f + \sum_i \int_{L_i} f - \sum_i \int_{\gamma_i} f - \sum_i \int_{L_i} f = \int_\gamma - \sum_i \int_{\gamma_i} f :$$

$$\int_\gamma f = \sum_i \int_{\gamma_i} f.$$

In $\int_{\gamma_i} f$, the holomorphic part gives 0, by Cauchy's Theorem. If the singular part is $\sum_{-\infty}^{-1} c_n(z-z_i)^n$ (so c_{-1}, or $c_{-1,i}$, is the residue at z_i),

$$\int_{\gamma_i} f = \int_{\gamma_i} \sum_{-\infty}^{-1} c_n(z-z_i)^n \, dz = \sum_{-\infty}^{-1} c_n \int_{\gamma_i} (z-z_i)^n \, dz$$

(interchanging $\int$ and $\sum$ by *uniform* convergence, as in the Proof of Laurent's Theorem). By the Fundamental Integral (II.4, Lecture 16), the integral above is $2\pi i$ if $n = -1$ and 0 otherwise. So $\int_\gamma f = \sum_i 2\pi i c_{-1} = 2\pi i \sum_i Res f$, summed over singularites *inside* γ. //

12. Finding residues
Simple Pole of f at α: $f(z) = g(z)/(z-\alpha)$, g holomorphic at α, $g(\alpha) \neq 0$. Cauchy-Taylor Theorem: $g(z) = \sum_0^\infty c_n(z-\alpha)^n$ ($c_0 \neq 0$). So

$$f(z) = \frac{g(z)}{z-\alpha} = \frac{c_0}{z-\alpha} + c_1 + c_2(z-\alpha) + \ldots :$$

$Res_\alpha f$ = coefficient of $1/(z-\alpha)$ on RHS $= c_0 = g(\alpha)$. So we get the *Cover-up Rule.*

$$Res_\alpha f = g(\alpha) :$$

cover up $1/(z-\alpha)$, then *substitute* $z = \alpha$.
Multiple Poles. (a) If $z = \alpha$ is a multiple pole, put $z = \alpha + \zeta$ (ζ small), and power-series expand in powers of ζ: Res_α = coefficient of $1/\zeta$.
(b) *Derivative Rule.* If α is a pole of f of order m, $f(z) = g(z)/(z-\alpha)^m$ (g holomorphic at α, $g(\alpha) \neq 0$). Cauchy-Taylor Theorem: $g(z) = \sum_0^\infty c_n(z-\alpha)^n$, $c_n = g^{(n)}(\alpha)/n!$. So taking $n = m-1$ gives

$$Res_\alpha f = g^{(m-1)}(\alpha)/(m-1)!$$

Note. 1. Do *not* use the Cover-Up rule for multiple poles!
2. Sometimes, we know the pole is simple (even though this may not be immediately obvious) – e.g., when the Laurent expansion only contains a $1/\zeta$). We may then find the residue by using the Cover-Up Rule.
Example (Ch. III). To find the residue at $i\pi/2$ of $f(z) := z^2e^z/(1+e^{2z})$ (poles where $e^{2z} = e^{(2n+1)i\pi}$, $z = (n+\frac{1}{2})i\pi$, all simple).

$$f(z) := z^2e^z/(1+e^{2z}) = g(z)/(z - i\pi/2), \qquad g(z) = \frac{(z-i\pi/2)}{(1+e^{2z})}.z^2e^z.$$

Using the Cover-Up Rule, we find $Res_{i\pi/2}\, f = g(i\pi/2)$. The z^2e^z factor is $(i\pi/2)^2.e^{i\pi/2} = -i\pi^2/4$ at $i\pi/2$. The fraction on the right is an indeterminate form at $i\pi/2$, which we can evaluate by
(a) L'Hospital's Rule (just as in Real Analysis):

$$\frac{(z-i\pi/2)}{(1+e^{2z})} \sim \frac{1}{2e^{2z}} \to -1/2 \qquad (z \to i\pi/2),$$

giving the residue as $(-1/2).(-i\pi^2/4) = i\pi^2/8$, as before, or
(b) power-series expansion: with $z = i\pi/2 + \zeta$, the fraction is $z/[1-(1+2\zeta+\cdots)] \to -1/2$ ($z \to i\pi/2$), again as before.

You may wish to compare these two ways of finding the residue in this example. If you have a preference, this may guide you more generally.