m2pm3l27(11).tex
Lecture 27. 8.3.2011.

Combining,

$$\int_{\gamma_3} f \to \int_{\infty}^{-\infty} e^{-x^2/2}.e^{y^2/2}.e^{-ixy}dx = -\sqrt{2\pi} \qquad (R \to \infty).$$

So (dividing by $\sqrt{2\pi}$ and by $e^{y^2/2}$, and reversing the direction of integration)

$$\int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}}.e^{-ixy}dx = e^{-y^2/2}.$$

The RHS is real, so the LHS is real. Take complex conjugates:

$$\int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}}.e^{ixy}dx = e^{-y^2/2}.$$

This gives the characteristic function (CF) of the standard normal density $\phi(x) := e^{-x^2/2}/\sqrt{2\pi}$, as given in Lecture 1 (the CF is the *Fourier transform* of a probability density).

Example. We give a further example of translation of the line of integration.

$$I := \int_{-\infty}^{\infty} \frac{u^2 e^u}{1+e^{2u}}\,du = \frac{\pi^3}{8}, \quad \int_{-\infty}^{\infty} \frac{ue^u}{1+e^{2u}}\,du = 0.$$

Take $f(z) := z^2 e^z/(1+e^{2z})$: f has a singularity where $1+e^{2z} = 0$, $e^{2z} = -1 = e^{(2n+1)i\pi}$, $z_n = (n+1/2)i\pi$. For γ, take the rectangle with vertices $\pm R$, $\pm R + i\pi$, with $\gamma_1 := [-R, R]$, $\gamma_2 := [R, R+i\pi]$, $\gamma_3 := [R+i\pi, -R+i\pi]$, $\gamma_4 := [-R+i\pi, -R]$. The only singularity of f inside γ comes from $n = 0$, at $z = i\pi/2$.

To find the residue of f at this pole, put $z = \frac{i\pi}{2} + \zeta$: as $e^{i\pi/2} = i$, $e^{i\pi} = -1$,

$$f(z) = \frac{(\frac{i\pi}{2}+\zeta)^2.i.e^{\zeta}}{1-e^{2\zeta}} = \frac{\frac{-i\pi^2}{4}\left(1+\frac{2\zeta}{i\pi}\right)^2(1+\zeta+\frac{1}{2}\zeta^2+...)}{1-[1+2\zeta+\frac{4}{2}\zeta^2+...]} = \frac{\frac{i\pi^2}{4}\left(1-\frac{4i\zeta}{\pi}...\right)(1+\zeta...)}{2\zeta(1+\zeta+...)}$$

$$= \frac{i\pi^2}{8\zeta}(1-\frac{4i\zeta}{\pi}...)(1+\zeta...)(1-\zeta...).$$

So $Res_{i\pi/2} f$ = coefficient of $1/\zeta$ on RHS $= i\pi^2/8$.
The contributions along γ_2, $\gamma_4 \to 0$, $(R \to \infty)$ (exponentially fast).

$$\int_{\gamma_1} = \int_{-R}^{R} \to \int_{-\infty}^{\infty} \frac{u^2 e^u}{1+e^{2u}}\, du = I.$$

$$\int_{\gamma_3} = \int_{R}^{-R} \frac{(u+i\pi)^2.(-)e^u}{1+e^{2u}}\, du = \int_{-R}^{R} \frac{(u^2+2i\pi u-\pi^2)e^u}{1+e^{2u}}\, du$$

$$\to \int_{-\infty}^{\infty} \frac{u^2 e^u}{1+e^{2u}}\, du + 2i\pi \int_{-\infty}^{\infty} \frac{ue^u}{1+e^{2u}}\, du - \pi^2 \int_{-\infty}^{\infty} \frac{e^u}{1+e^{2u}}\, du.$$

Now

$$\int_{-\infty}^{\infty} \frac{ue^u}{1+e^{2u}}\, du = \int_{-\infty}^{\infty} \frac{u}{e^{-u}+e^u}\, du = 0$$

(odd integrand between symmetrical limits), and (substituting $t := e^u$)

$$\int_{-\infty}^{\infty} \frac{e^u}{1+e^{2u}}\, du = \int_{-\infty}^{\infty} \frac{d(e^u)}{1+e^{2u}} = \int_{0}^{\infty} \frac{1}{1+t^2}\, dt = [\tan^{-1} t]_0^\infty = \frac{\pi}{2}.$$

Combining,

$$\int_{\gamma_3} f = I + 2i\pi.0 - \pi^2.\frac{\pi}{2} = I - \frac{\pi^3}{2}.$$

(The fact that $\int \gamma_3 f$ involves the answer I motivates the choice of the vertices $\pm R + i\pi$. One can with hindsight see this in the form of f, since $(e^{u+i\pi})^2 = e^{2\pi i}.e^{2u} = e^{2u}$.)

By CRT: $\int_\gamma f = 2\pi i Res_{\ i\pi/2}\ f$:

$$I + (I - \frac{\pi^3}{2}) = 2\pi i.i\pi^2/8 = -\frac{\pi^3}{4}: \qquad 2I = \frac{\pi^3}{2} - \frac{\pi^3}{4} = \frac{\pi^3}{4}: \qquad I = \pi^3/8.//$$