m2pm3l3.tex

Lecture 3. 13.1.2011.

Cartesians v. Polars.

For addition and subtraction, cartesians are convenient: Re and Im add and subtract nicely. For multiplication and division, polars are convenient:

$$z_1 z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = (r_1 r_2) e^{i(\theta_1+\theta_2)},$$

$$\frac{z_1}{z_2} = \frac{r_i e^{i\theta_1}}{r_i e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}.$$

What is i?

We first meet i as a formal square root of -1: $i = \sqrt{-1}$.
Then all expressions $i \times i = i^2$ are replaced by -1: $i^2 = -1$.
Rotation.

In the Argand diagram, i is $(0, 1) = 0 \times 1 + 1 \times i$ in Cartesians, $1 \times e^{i\frac{\pi}{2}}$ in polars. As $\cos(\pi/2) = 0$, $\sin(\pi/2) = 1$,

$$iz = \left(1 e^{i\pi/2}\right)\left(r e^{i\theta}\right) = r e^{i(\theta+\pi/2)}.$$

So multiplying by i rotates the radius vector through a right-angle anticlockwise: "i is the order Left Turn" (not a number so much as an *operation*).
2×2 *matrices*:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}: \qquad I^2 = I.$$

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}: \quad J^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I.$$

We can if we wish regard the Argand representation as

$$(x, y) \longleftrightarrow xI + yJ,$$

and work with 2×2 real matrices.
Note The reals $\mathbf{R}$ form a *field* in the language of Algebra. This field is not *algebraically closed*: a real polynomial need not have real roots: (e.g. $x^2 + 1$). By adjoining to the field $\mathbf{R}$ the element i (or i and $-i$), we obtain a bigger field, $\mathbf{C}$, in which $x^2 + 1$ *does* have roots. This passage from $\mathbf{R}$ to $\mathbf{C}$ is called *field extension*. It leads to the subject of *Galois theory* (Evariste GALOIS

(1811-1832), in 1832). We do not need to go further. Now all complex polynomials of degree n have n complex roots (counted according to multiplicity): this is the *Fundamental Theorem of Algebra*, which we shall prove later (II.6).

Despite the name, the Fundamental Theorem of Algebra is a result not of Algebra but of Analysis. Its proof needs limiting operations. We can think of Analysis as the subject concerning limit operations (such as convergence, differentiation, integration etc.). But basically we are doing Analysis when we are making essential use of the properties of the real or complex number systems, **R** or **C** (G. H. Hardy (1877-1947) used to say that an analyst was a mathematician habitually seen in the company of the real or complex number systems).

Theorem (Triangle Inequality).

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

Proof.

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)\overline{(z_1 + z_2)} \\
&= (z_1 + z_2)(\overline{z_1} + \overline{z_2}) \\
&= z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2} \\
&= |z_1|^2 + |z_2|^2 + 2Re(z_1 z_2) \quad (z_2\overline{z_1} = \overline{z_1\overline{z_2}}; 2Re(z) = z + \overline{z}) \\
&\leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \quad (Re(z) = x \leq \sqrt{x^2 + y^2} = |z|; |z_1 z_2| = |z_1||z_2|) \\
&= (|z_1| + |z_2|)^2. \quad //
\end{aligned}$$

Cor. Equality holds, $|z_1 + z_2| = |z_1| + |z_2|$, iff z_1, z_2 have the same argument – i.e. lie on the same ray $arg(z) = \theta$.

Proof. Equality holds iff $Re(z_1\overline{z_2}) = |z_1 z_2|$. But $z_1\overline{z_2} = r_1 e^{i\theta_1} \times r_2 e^{-i\theta_2} = r_1 r_2 e^{i(\theta_1 - \theta_2)} = r_1 r_2 = |z_1 z_2|$ iff $\theta_1 = \theta_2$. //

Note [Geometrical Interpretation].
Sum of lengths of two sides of a triangle $\geq$ lengths of 3rd side. Equality iff triangle degenerates to a line.

Note [Physical Interpretation].
Vector addition. Triangle of Forces.