m2pm3l30(11).tex
Lecture 30. 17.3.2011.

$f(z) = z^{t-1}e^{-z}$ (The origin is a branch-point). Take $0 < t < 1$, $z = Re^{i\theta}$.

$$\begin{aligned}
\left|\int_{\gamma_2} f\right| &= \left|\int_0^{\frac{\pi}{2}} R^{t-1}e^{i(t-1)\phi}e^{-R\cos\theta}e^{-R\sin\theta}iRe^{i\theta}\,d\theta\right| \\
&\le R^t\int_0^{\frac{\pi}{2}} e^{-R\cos\theta}\,d\theta \quad (\phi := \frac{\pi}{2}-\theta) \\
&= R^t\int_0^{\frac{\pi}{2}} e^{-R\sin\phi}\,d\phi \le R^t\int_0^{\frac{\pi}{2}} e^{-2R\frac{\phi}{\pi}}\,d\phi \quad \text{(Lemma)} \\
&= R^t\frac{\pi}{2R}(1-e^{-R}) \le \frac{\pi}{2R^{1-t}} \to 0 \qquad (R\to\infty).
\end{aligned}$$

On γ_4, $e^{-z} = e^{-re^{i\theta}} \sim 1$ $(r \to 0)$. So by ML,

$$\left|\int_{\gamma_4} f\right| \le O(r^{t-1}).O(r) = O(r^t) \to 0 \quad (r\to 0) \qquad (t>0).$$

Now $\int_{\gamma_1} f \to \int_0^\infty x^{t-1}e^{-x}\,dx = \Gamma(t)$, and on γ_3, $z = iy = e^{i\pi/2}y$, so

$$\int_{\gamma_3} f \to -\int_0^\infty (iy)^{t-1}e^{-iy}i\,dy = -e^{it\pi/2}\int_0^\infty y^{t-1}(\cos y - i\sin y)\,dy.$$

Cauchy's Theorem: $\int_\gamma f = 0$. So as $\int_{\gamma_1} f \to \Gamma(t)$,

$$\Gamma(t) = e^{it\pi/2}\int_0^\infty y^{t-1}(\cos y - i\sin y)\,dy : \qquad \int_0^\infty x^{t-1}(\cos x - i\sin x)\,dx = e^{-it\pi/2}\Gamma(t).$$

Hence we can take the real and imaginary parts:

$$\int_0^\infty x^{t-1}\cos x\,dx = \cos\frac{1}{2}\pi t\Gamma(t), \qquad \int_0^\infty x^{t-1}\sin x\,dx = \sin\frac{1}{2}\pi t\Gamma(t).$$

Mellin Transforms. If $f : \mathbf{R}_+ \to \mathbf{R}$, $\tilde{f}(s) := \int_0^\infty x^{s-1}f(x)\,dx$ is called the Mellin transform of f.
Examples.

f	$\tilde{f}$	(where)
e^{-x}	$\Gamma(s)$	(definition of Γ)
$\cos x$	$\cos\frac{1}{2}\pi s\Gamma(s)$	(above)
$\sin x$	$\sin\frac{1}{2}\pi s\Gamma(s)$	(above)
$1/(1+x)$	$\pi/\sin\pi s = \Gamma(s)\Gamma(1-s)$	(below)
$1/(e^x-1)$	$\Gamma(s)\zeta(s)$	(below)

Proof.

$$\begin{aligned}\int_0^\infty \frac{x^{s-1}}{e^x-1}\,dx &= \int_0^\infty x^{s-1}\frac{e^{-x}}{1-e^{-x}}\,dx = \int_0^\infty x^{s-1}e^{-x}\sum_{n=0}^\infty e^{-nx}\,dx = \int_0^\infty x^{s-1}\sum_{n=0}^\infty e^{-(n+1)x}\,dx \\ &= \sum_{n=0}^\infty \int_0^\infty x^{s-1}e^{-(n+1)x}\,dx \quad \text{(OK by monotone convergence)} \\ &= \sum_{n=1}^\infty \int_0^\infty x^{s-1}e^{-nx}\,dx = \sum_{n=1}^\infty \int_0^\infty u^{s-1}e^{-u}\,du = \zeta(s)\Gamma(s).\end{aligned}$$

Note. Mellin transforms $\leftrightarrow (\mathbf{R}_+, \times)$ (multiplicative group of positive reals) as Fourier/Laplace transforms $\leftrightarrow (\mathbf{R}, +)$ (additive group of reals.)

6. Integrals involving many-valued functions
Example.

$$I = \int_0^\infty \frac{x^{a-1}}{1+x}\,dx = \frac{\pi}{\sin \pi a} \qquad (0 < a < 1).$$

Take $f(z) := z^{a-1}/(1+z)$, γ the 'keyhole' contour consisting of: γ_1: $r \le x \le R$, $\arg z = 0$, upper edge of the cut; γ_2: circle, radius R, +ve sense; γ_3: lower edge of cut, -ve sense; γ_4: circle, radius r, -ve sense.

$$\int_{\gamma_1} f \to I \quad (r \to 0,\ R \to \infty).$$

On γ_2: $1/(1+z) \sim 1/z = O(1/R)$. By ML, as on γ_2 $1/(1+z) = O(1/R)$, $z^{a-1} = O(R^{a-1})$, and the length $L(\gamma_2) = O(R)$,

$$\left|\int_{\gamma_2} f\right| = O(1/R).O(R^{a-1}).O(R) = O(R^{a-1}) \to 0 \qquad (R \to \infty) \qquad (a < 1).$$

On γ_3: $\arg z = 2\pi$, $z = xe^{2\pi i}$, $z^{a-1} = x^{a-1}.e^{2\pi a i}I$. So

$$\int_{\gamma_3} f \to \int_\infty^0 \frac{x^{a-1}e^{2\pi a i}}{1+x}\,dx = -e^{2\pi a i}\int_0^\infty \frac{x^{a-1}}{1+x}\,dx = -e^{2\pi a i}I.$$

On γ_4: $f(z) \sim z^{a-1} = O(r^{a-1})$. By ML,

$$\left|\int_{\gamma_4} f\right| = O(r^{a-1}).O(r) = O(r^a) \to 0 \quad (r \to 0) \quad (a > 0).$$