m2pm3l31(11).tex
Lecture 31. 21.3.2011.

f has a pole at $z = -1 = e^{i\pi}$, of residue $e^{i\pi(a-1)} = -e^{i\pi a}$. By CRT:

$$I - Ie^{2\pi ia} = -2\pi ie^{i\pi a},$$

$$I = \pi.\frac{-2ie^{i\pi a}}{1 - e^{2\pi ia}} = \pi.\frac{-2i}{e^{-i\pi a} - e^{i\pi a}} = \pi.\frac{2i}{e^{i\pi a} - e^{-i\pi a}} = \frac{\pi}{\sin \pi a}.$$

So (Euler's reflection formula for the Gamma function – II.8.4) $I = \Gamma(a)\Gamma(1-a)$.

Note. For $0 < x < 1$, $\Gamma(x)\Gamma(1-x) = \int_0^\infty v^{x-1}dv/(1+v)$ (Problems 4 Q4).
Proof:

$$\Gamma(x)\Gamma(1-x) = \int_0^\infty t^{x-1}e^{-t}\,dt.\int_0^\infty u^{-x}e^{-u}\,du.$$

Writing $u = tv$ and interchanging the order of integrations, this is

$$\int_0^\infty v^{-x}\,dv\int_0^\infty e^{-t(1+v)}\,dt = \int_0^\infty \frac{v^{-x}}{1+v}\,dv = \int_0^\infty \frac{v^{x-1}}{1+v}\,dv$$

($x \to 1-x$ leaves LHS unchanged, so also RHS).

7. Summation of Series

Consider the function $\cot \pi z = \cos \pi z/ \sin \pi z$, with simple poles at $z = n$. For $z = n + \zeta$, ζ small,

$$\cot \pi z = \frac{\cos(n\pi + \pi\zeta)}{\sin(n\pi + \pi\zeta)} = \frac{(-1)^n \cos \pi\zeta}{(-1)^n \sin \pi\zeta} \sim \frac{1}{\pi\zeta} \quad (\zeta \to 0): \quad Res_n \cot \pi z = \frac{1}{\pi}.$$

Also, *cosec* $\pi z = 1/ \sin \pi z$ has simple poles at $z = n$, and

$$Res_n \text{cosec } \pi z = \frac{(-1)^n}{\pi}.$$

If $f(z)$ is holomorphic at $z = n$, by the Cover-Up Rule,

$$Res_n\ f(z) \cot \pi z = \frac{f(n)}{\pi}, \quad Res_n\ f(z)\text{cosec } \pi z = \frac{(-1)^n f(n)}{\pi}.$$

This suggests a method of summing series $\sum f(z)$ or $\sum(-1)^n f(n)$ by CRT. We need suitable contours. We quote (for proof – not examinable – see Website):

Lemma (Squares Lemma). Let C_N be a square with vertices $(N+\frac{1}{2})(\pm 1 \pm i)$. Then $\operatorname{cosec} \pi z$, $\cot \pi z$ are *uniformly bounded* (in z and N) on C_N.

Example.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \qquad \text{(Euler)}.$$

Proof. Take $f(z) = 1/z^2$. Then $f(z) \cot \pi z$ has simple poles at $z = n \neq 0$ residue $f(n)/\pi = 1/(\pi n^2)$, and a triple pole at $z = 0$. Near 0,

$$f(z) \cot \pi z = \frac{\cos \pi z}{z^2 \sin \pi z} = \frac{1 - \frac{\pi^2 z^2}{2} + ...}{z^2 \left(\pi z - \frac{\pi^3 z^3}{6} + ...\right)}$$

$$= \frac{1}{\pi z^3}.(1 - \frac{\pi^2 z^2}{2} + ...)(1 + \frac{\pi^2 z^2}{6} - ...) = \frac{1}{\pi z^3}.(1 - \frac{\pi^2 z^2}{3} + ...).$$

The residue is the coefficient of $1/z$, so

$$Res_0 \; f(z) \cot \pi z = -\pi/3.$$

Take $f(z) = 1/z^2$, squares C_N as in the Lemma. By CRT:

$$\left| \int_{C_N} \frac{\cot \pi z}{z^2} \, dz \right| = 2\pi i \sum Res = 2\pi i \left(-\pi/3 + \sum_{n=-N}^{N} \frac{1}{\pi n^2} \right) \quad \text{(Cover-Up Rule)}.$$

By ML: as $\cot \pi z$ is bounded ($= O(1)$) on the C_N, $1/z^2 = O(1/N^2)$ on the C_N, and the C_N have length $O(N)$,

$$\left| \int_{C_N} \frac{\cot \pi z}{z^2} \, dz \right| = O(1).O(1/N^2).O(N) = O(1/N) \to 0 \quad (N \to \infty).$$

Combining, we get

$$-\frac{\pi}{3} + \frac{2}{\pi} \sum_{n=1}^{N} 1/n^2 \to 0 \quad (N \to \infty): \quad \zeta(2) := \sum_{n=1}^{\infty} 1/n^2 = \pi^2/6. \qquad //$$

Carrying the expansion one stage further, we get similarly

$$\zeta(4) = \sum_{n=1}^{\infty} 1/n^4 = \pi^4/90.$$