m2pm3l32(11).tex
Lecture 32. 22.3.2011.

$$\zeta(4) = \sum_{n=1}^{\infty} 1/n^4 = \pi^4/90.$$

Proof.

$$\frac{\cot \pi z}{z^4} = \frac{\cos \pi z}{z^4 \sin \pi z} = \frac{1 - \frac{\pi^2 z^2}{2} + \frac{\pi^4 z^4}{24} ...}{z^4 \left(\pi z - \frac{\pi^3 z^3}{6} + \frac{\pi^5 z^5}{120} ...\right)}$$

$$= \frac{1}{\pi z^5}\left(1 - \frac{\pi^2 z^2}{2} + \frac{\pi^4 z^4}{24} ...\right).(1 - \frac{\pi^2 z^2}{6} + \frac{\pi^4 z^4}{120} ...)^{-1}$$

$$= \frac{1}{\pi z^5}\left(1 - \frac{\pi^2 z^2}{2} + \frac{\pi^4 z^4}{24} ...\right).(1 - \{...\})^{-1},$$

say. Since $(1 - \{...\})^{-1} = 1 + \{...\} + \{...\}^2 + ...$, the last factor on the RHS is

$$1 + \frac{\pi^2 z^2}{6} - \frac{\pi^4 z^4}{120} + \frac{\pi^4 z^4}{36} + ... = 1 + \frac{\pi^2 z^2}{6} + \pi^4 z^4 (\frac{1}{36} - \frac{1}{120}) = 1 + \frac{\pi^2 z^2}{6} + \pi^4 z^4 \frac{7}{360},$$

neglecting terms beyond z^4. So multiplying up the last two brackets on the RHS, we get three terms in z^4, each of which will contribute to the residue (coefficient of $1/z$), in view of the z^{-5} factor. This gives

$$Res_0 \frac{\cot \pi z}{z^4} = \text{ coefficients of } 1/z \text{ on RHS } = \frac{1}{\pi}.\pi^4 \left(\frac{7}{360} + \frac{1}{24} - \frac{1}{12}\right) = ... = -\frac{\pi^3}{45}.$$

As before:

$$-\frac{\pi^3}{45} + \frac{2}{\pi}\sum_{n=1}^{N} 1/\pi^4 \to 0 \quad (N \to \infty): \quad \zeta(4) = \sum_{1}^{\infty} 1/n^4 = \pi^4/90. \qquad //$$

8. Expansion of a meromorphic function.
Example.

$$f(z) = cosec\ z - \frac{1}{z} = \frac{1}{\sin z} - \frac{1}{z}.$$

Simple pole at $z = n\pi$, $n \neq 0$. For $z = n\pi + \zeta$,

$$f(z) = \frac{1}{\sin(n\pi + \zeta)} - \frac{1}{n\pi + \zeta} = \frac{(-1)^n}{\sin \zeta} - \frac{1}{n\pi + \zeta} \sim \frac{(-1)^n}{\zeta} \quad (\zeta \to 0):$$

$$Res_{n\pi} f = (-1)^n, \quad n \neq 0.$$

At $n = 0$,

$$f(z) = \frac{z - \sin z}{z \sin z} = \frac{z - \left(z - \frac{z^3}{6} \ldots\right)}{z\left(z - \frac{\pi^3}{6} \ldots\right)} = \frac{\frac{1}{6}z^3 + \ldots}{z^2\left(1 - \frac{z^2}{6} \ldots\right)} \sim \frac{z}{6} \to 0 \quad (z \to 0) :$$

no singularity, so no residue.

By the Squares Lemma (on *cosec* πz, on C_N): *cosec* z is bounded on the squares Γ_N with vertices $(N + \frac{1}{2})\pi(\pm 1 \pm i)$. Consider

$$I_N(z) := \int_{\Gamma_N} \frac{f(w)}{w(w-z)}\, dw = \int_{\Gamma_N} \frac{cosec\ w - 1/w}{w(w-z)}\, dw.$$

$$\left| \int_{\Gamma_N} \frac{cosec\ w}{w(w-z)}\, dw \right| = O(1).O(1/N^2).O(N) = O(1/N) \to 0 \quad (N \to \infty)$$

$$\left| \int_{\Gamma_N} \frac{1/w}{w(w-z)}\, dw \right| = O(1/N).O(1/N^2).O(N) = O(1/N^2) \to 0.$$

So $I_N(z) \to 0$ as $N \to \infty$. By CRT:

$$I_N(z) = 2\pi i \sum Res \frac{cosec\ w - 1/w}{w(w-z)},$$

over the singularities inside Γ_N. For fixed z, z is *inside* Γ_N for large enough N. By the Cover-Up rule:

$$Res_z = \frac{cosec\ z - 1/z}{z},$$

while proceeding as above $(cosec\ w - 1/w)/(w(w-z))$ has no singularity at $z = 0$. For $n \neq 0$,

$$Res_{n\pi} = \frac{(-1)^n}{n\pi(n\pi - z)} = \frac{(-1)^n}{(-z)}\left(\frac{1}{n\pi} - \frac{1}{n\pi - z}\right) = -\frac{(-1)^n}{z}\left(\frac{1}{z - n\pi} + \frac{1}{n\pi}\right).$$

So

$$I_N(z) = 2\pi i \left\{ -\sum_{n=-N\, n\neq 0}^{N} \left[\frac{(-1)^n}{z}\left(\frac{1}{z - n\pi} + \frac{1}{n\pi}\right)\right] + \frac{cosec\ z - 1/z}{z} \right\} \to 0.$$

We can cancel $2\pi i/z$. Then replace $\sum_{n=-N\, n\neq 0}^{N} = \sum_{-N}^{-1} + \sum_{1}^{N}$ by $\sum_{1}^{N}\{...+...\}$: the $1/(n\pi)$ and $-1/(n\pi)$ cancel, and

$$\frac{1}{z-n\pi} + \frac{1}{z+n\pi} = \frac{2z}{z^2-n^2\pi^2}.$$

We obtain

$$\mathit{cosec}\ z = \frac{1}{z} + 2z\sum_{n=1}^{\infty}\frac{(-)^n}{z^2-n^2\pi^2} = \frac{1}{z} + 2z\sum_{even} - 2z\sum_{odd}. \qquad (i)$$

Similarly,

$$\mathit{cot}\ z = \frac{1}{z} + 2z\sum_{n=1}^{\infty}\frac{1}{z^2-n^2\pi^2}. \qquad (ii)$$

In III.9, *Infinite products for* $\sin, \cos, \tan$ (Website – not examinable), we start with (i) and (ii), integrate, and obtain the infinite products for sin, cos and tan. These give extensions to entire functions of the Fundamental Theorem of Algebra, displaying a polynomial as a product of linear factors vanishing at its roots. From these, we can again obtain $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, and Wallis's product for π (this time by Complex Analysis).