m2pm3l33(11).tex

## Lecture 33. 24.3.2011

Contours for Chapter III.

- 1. Circles (III.1, Lecture 26);
- 2. Ellipses (III.1, Lecture 26, Problems 6 Q5);
- 3. Squares (III.7, III.8, Lecture 31, 32);
- 4. Rectangles (III.2, Lecture 26, 27);
- 5. Semicircles (III.3, Lecture 28);
- 6. Sectors (III.10, Lecture 33);
- 7. Indented semicircle, (III.4, Lecture 28);
- 8. Indented sector (III.5, Lecture 29-30);
- 9. Keyhole contour (III.6, Lecture 30).

## 10. Further examples.

1. Consider

$$I_n := \int_0^\infty \frac{1}{1+x^n} \, dx = \frac{\pi}{n \sin \pi/n} \qquad (n = 2, 3, \ldots).$$

First Proof: sector contour. Let  $f(z) := 1/(1 + z^n)$ , and take the contour as a sector, with  $\gamma_1 = [0, R]$ ,  $\gamma_2$  on the arc |z| = R,  $0 \le \arg z \le 2\pi/n$ , and  $\gamma_3$  the path back to the origin.

By ML,  $\int_{\gamma_2} f = O(R^{-n}.R) \to 0$  as  $R \to \infty$ .

$$\int_{\gamma_1} f \to I \qquad (R \to \infty).$$

On  $\gamma_3$ , z goes from R to O,  $z = xe^{2\pi i/n}$ ,  $dz = e^{2\pi i/n} dx$ . So  $\int_{\gamma_3} f \to -e^{2\pi i/n} I$ . So

$$\int_{\gamma} f \to I\left(1 - e^{2\pi i/n}\right).$$

By CRT:

$$\int_{\gamma} f = 2\pi i \sum Resf,$$

and f singular where  $z^n = -1 = e^{i\pi} = e^{(2k+1)\pi}$ ,  $z = e^{i\pi(2k+1)/n}$ . Only k = 0,  $z = e^{i\pi/n}$  is inside  $\gamma$ . So we have a simple pole:

$$f(z) = \frac{z - e^{i\pi/n}}{(1+z^n)(z - e^{i\pi/n})}.$$

Using the Cover-Up Rule and L'Hospital's Rule (II.12, Lecture 25),

$$Res_{e^{i\pi/n}}f = \lim_{z \to e^{i\pi/n}} \frac{z - e^{i\pi/n}}{z^n + 1} = \lim_{z \to e^{i\pi/n}} \frac{1}{nz^{n-1}} = \frac{1}{ne^{i\pi(n-1)/n}} = \frac{e^{i\pi/n}}{ne^{i\pi}} = -\frac{e^{i\pi/n}}{n}.$$

Combining, and using Euler's formula:

$$I = \frac{2\pi i \cdot (-1)^n e^{i\pi/n}}{n(1 - e^{2\pi i/n})} = \frac{\pi}{n} \cdot \frac{-2i}{e^{i\pi/n} - e^{i\pi/n}} = \frac{\pi}{n \sin \pi/n}.$$
 //

We note two special cases:

$$I_2 = \int_0^\infty \frac{1}{1+x^2} \, dx = \left[ \tan^{-1} x \right]_0^{\pi/2} = \frac{\pi}{2}; \qquad I_6 = \int_0^\infty \frac{1}{1+x^6} = \frac{\pi}{6\sin\pi/6} = \frac{\pi}{3}.$$

Second Proof: by the integral of III.6 (keyhole contour). Put  $x^n = y$ ,  $x = y^{1/n}$ ,  $dx = (1/n)y^{(1/n)-1}dy$ .

$$I = \int_0^\infty \frac{1}{n} \cdot \frac{y^{(1/n)-1} dy}{1+y} = \frac{\pi}{n \sin(\pi/n)}.$$

2. Consider

$$I := \int_0^\infty \frac{x^{1/2}}{1+x^3} \, dx = \frac{\pi}{3}$$

First Proof: Real Analysis. Put  $y := x^{3/2}$ ,  $x = y^{2/3}$ .  $dx = \frac{2}{3}y^{-1/3} dy$ . Then

$$I = \int_0^\infty \frac{y^{\frac{1}{3}} \frac{2}{3} y^{-\frac{1}{3}}}{1+y^2} \, dy = \int_0^\infty \frac{2}{3} \frac{1}{1+y} \, dy = \frac{2}{3} \left[ \tan^{-1} y \right]_0^\infty = \frac{\pi}{3}.$$

Second Proof: Complex Analysis (we give this for practice in handling manyvalued integrands, and in calculating residues). Let  $f(z) = \frac{z^{1/2}}{(1 + z^3)}$ (double valued, because of the square root – as in Real Analysis!). By ML:

$$\left| \int_{\gamma_2} f \right| = O\left(\frac{R^{1/2}}{R^3} \cdot R\right) \to 0 \qquad (R \to \infty), \qquad \left| \int_{\gamma_4} f \right| = O\left(r^{1/2} \cdot r\right) \to 0 \qquad (r \to 0),$$
$$\int_{\gamma_3} f \to I \qquad (r \to 0, \ r \to \infty).$$

On  $\gamma_3$ ,  $z = xe^{2\pi i}$ ,  $z^{1/2} = x^{1/2}e^{i\pi} = -x^{1/2}$ . So

$$\int_{\gamma_3} f \to \int_{\infty}^0 \frac{-x^{1/2}}{1+x^3} \, dx = I, \qquad \int_{\gamma} f \to 2I.$$

By CRT:

$$\int_{\gamma} f = 2\pi i \sum Resf,$$

and f has singularities where  $1 + z^3 = 0$ ,  $z^3 = -1 = e^{i\pi} = e^{(2k+1)i\pi}$ ,  $z = e^{(2k+1)i\pi/3}$ , giving (for k = 0, 1, 2)  $z = e^{i\pi/3}$ ,  $e^{i\pi} = -1$ ,  $e^{5i\pi/3}$ . So

$$f(z) = \frac{z^{1/2}}{(z - e^{i\pi/3})(z+1)(z - e^{5i\pi/3})}.$$

Cover-Up Rule:

$$\begin{split} Res_{e^{i\pi/3}}f &= \frac{e^{i\pi/6}}{(e^{i\pi/3}+1)(e^{i\pi/3}-e^{5i\pi/3})} = e^{i\pi/6}/[e^{2i\pi/3}-1+e^{i\pi/3}-e^{5i\pi/3}] \\ &= e^{i\pi/6}/[(-\frac{1}{2}+i\frac{\sqrt{3}}{2})-1+(\frac{1}{2}+i\frac{\sqrt{3}}{2})-(\frac{1}{2}-i\frac{\sqrt{3}}{2})] \\ &= e^{i\pi/6}/[-\frac{3}{2}+3.\frac{i\sqrt{3}}{2}] = e^{i\pi/6}/[3.e^{2\pi i/3}] = e^{-i\pi/2}/3 = -i/3. \\ Res_{e^{i\pi}}f &= \frac{e^{i\pi/2}}{(-1-e^{i\pi/3})(-1-e^{5i\pi/3})} = i/[2+e^{i\pi/3}+e^{5i\pi/3}] \\ &= i/[2+(\frac{1}{2}+\frac{i\sqrt{3}}{2})+(\frac{1}{2}-\frac{i\sqrt{3}}{2})] = i/3. \\ Res_{e^{5i\pi/3}}f &= \frac{e^{5i\pi/6}}{(e^{5i\pi/3}+1)(e^{5i\pi/3}-e^{i\pi/3})} = e^{5i\pi/6}/[e^{4i\pi/3}-1+e^{5i\pi/3}-e^{i\pi/3}] \\ &= e^{5i\pi/6}/[(-\frac{1}{2}-i\frac{\sqrt{3}}{2})-1+(\frac{1}{2}-i\frac{\sqrt{3}}{2})-(\frac{1}{2}+i\frac{\sqrt{3}}{2})] \\ &= e^{5i\pi/6}/[[-\frac{3}{2}-3.\frac{i\sqrt{3}}{2}] = e^{5i\pi/6}/[-3.e^{i\pi/3}] = -e^{i\pi/2}/3 = i/3. \\ \text{So} \sum Res f = -i/3, 2\pi i \sum Res f = 2\pi/3; 2I = 2\pi/3, I = \pi/3. // \end{split}$$

To recapitulate on the exam: Q1: bits and pieces, from anywhere;

Q2,3: theory, Ch. II;

Q4(i),(ii): applications, Ch. III.

As always, I examine on what I have taught (and, usually, emphasized), aiming for a fair test of a working knowledge of the material. I do not expect candidates to leave the exam room feeling surprised.

Use the revision classes next term. Good luck with your revision, and the exam. NHB