m2pm3l4(11).tex
Lecture 4. 17.1.2011.

Euler's Formula (L. Euler (1707-1783)).

$$e^z = 1 + z + \frac{z^2}{2!} + ... + \frac{z^n}{n!}...,$$
$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - ...,$$
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - ...$$

Take $z = i\theta$, θ real:

$$\begin{aligned} e^{i\theta} &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} \\ &= (1 - \frac{z^2}{2!} + \frac{z^4}{4!} - ...) + i(z - \frac{z^3}{3!} + \frac{z^5}{5!} - ...) \\ &= \cos\theta + i\sin\theta. \end{aligned}$$

This is implicit in the *Argand representation*.
Note. Take $\theta = \pi$; then

$$e^{i\pi} = -1.$$

The extended complex plane $\mathbf{C}^*$.

R is totally ordered, so there are two directions in which to "go off to infinity", right to $+\infty$, and left to $-\infty$. We write $\mathbf{R}^* := \mathbf{R} \cup \{+\infty\} \cup \{-\infty\}$. What about **C**?

On **R**, recall graphs with asymptotes, e.g. $g(x) = 1/x$ or $g(x) = \tan(x)$. This suggests that, in some sense, '$+\infty$ and $-\infty$ are the same place'.

Stereographic Projection (G.F. RIEMANN (1826-66) in 1851), PTOLEMY, c. 160 AD).

Draw a picture of the unit sphere ('Earth'), showing the following:

Σ	Unit sphere
C	Unit circle in Oxy ("Equator")
N,S	North and South poles
GM	"Greenwich Meridian"

Line NP cuts Oxy-plane in $P^{'}$.

$$P \longrightarrow P^{'}$$

is called *stereographic projection.*

P Northern Hemisphere	$\longleftrightarrow$	$P^{'}$ outside unit circle C
P Southern Hemisphere	$\longleftrightarrow$	$P^{'}$ inside unit circle C
P on equator	$\longleftrightarrow$	$P^{'} = P$ on unit circle
P South Pole S	$\longleftrightarrow$	$P^{'}$ origin 0
P North Pole N	$\longleftrightarrow$	$P^{'}$?

Stereographic projection gives a 1-1 correspondence between the complex plane $\mathbf{C}$ and $\Sigma \setminus \{N\}$. Call this the *punctured sphere*

$$\Sigma' := \Sigma \setminus \{N\}.$$

We now complete Σ' to get Σ, by including the North Pole N. This corresponds (under stereographic projection) to completing $\mathbf{C}$ to get $\mathbf{C}^*$:

$$\mathbf{C}^* := \mathbf{C} \cup \{\infty\},$$

where ∞, the "point at infinity in $\mathbf{C}$", corresponds to "going off to infinity in all directions".

Note. 1. This is a special case of a general procedure, called *Alexandrov (one-point) compactification.*

2. See also the subject of *Projection Geometry* (Girard DESARGUES (1591-1661) in 1631) - the mathematics of perspective and computer graphics.

What is π?

From now on $\cos x$ and $\sin x$ are *defined* by their power-series expansions. So we should define π in terms of these also:

$$\pi := \text{smallest positive root of sin},$$

equivalently,

$$\pi/2 := \text{smallest positive root of cos}.$$

(This should be in all the books, but it isn't.) For background, see e.g.
E. F. WHITTAKER & G.N. WATSON, *Modern Analysis*, 4th ed. (1927/1946), CUP, Appendix.