m2pm3l5(11).tex
Lecture 5. 18.1.2011.

De Moivre's Theorem.

$$\cos n\theta + i\sin n\theta = e^{in\theta} = (e^{i\theta})^n = (cos\theta + i\sin\theta)^n = \sum_{k=0}^{n}\binom{n}{k}\cos^k\theta i^{n-k}\sin^{n-k}\theta.$$

Take real parts: writing c, s for $\cos\theta$, $\sin\theta$,

$$\cos n\theta = c^n - \binom{n}{2}c^{n-2}s^2 + \binom{n}{4}c^{n-4}s^4 + \ldots = c^n - \binom{n}{2}c^{n-2}(1-c^2) + \binom{n}{4}c^{n-4}(1-c^2)^2 + \ldots$$

So $\cos n\theta$ is a polynomial of degree n in $c = \cos\theta$ (Exam 2010, Q1).
Complements.

Draw a Venn diagram with two overlapping sets, showing their intersection and union. We only deal with subsets of a given fixed set, called the *universal set*, Ω (the 'frame in the Venn diagram'). In M2PM3 we take $\Omega = \mathbf{C}$, unless we say otherwise (e.g., $\Omega = \mathbf{C}^*$).
Recall De Morgan's Laws (Augustus De MORGAN (1806-1871), in 1870):

$$\begin{array}{rcll} (A\cup B)^c &=& A^c\cap B^c: & \text{Complement of union = Intersection of complements;} \\ (A\cap B)^c &=& A^c\cup B^c: & \text{Complements of intersection = Union of complements.} \end{array}$$

Similarly for arbitrary (e.g. infinite) unions and intersections.

2. Preliminaries from Real Analysis and Topology
1. *Conditional and Absolute Convergence.*
Recall: $\sum_0^\infty a_n$ *converges* means its partial sums $s_n := \sum_0^n a_k$ converge to a limit.
$\sum a_n$ *converges absolutely* if $\sum |a_n|$ converges.
Absolute convergence $\Longrightarrow$ convergence; the converse is false (e.g. $\sum_1^\infty (-1)^n/n$ converges, but $\sum_1^\infty 1/n$ diverges).
If $\sum a_n$ converges but not absolutely it is *conditionally convergent.*
Absolutely convergent series behave well under all operation – eg, rearrangement of the order of the terms. Conditionally convergent series do not, and must be handled with care.
2. *Uniform Convergence.*
Defn. A function $f : \mathbf{R} \to \mathbf{R}$ is *continuous* at x_0 if

$$\forall\epsilon > 0\,\exists\delta > 0 \text{ s.t. } \forall x \text{ with } |x - x_0| < \delta, \quad |f(x) - f(x_0)| < \epsilon.$$

Defn. A function $f : [a,b] \to \mathbf{R}$ is *uniformly continuous* on $[a,b]$ if

$$\forall \epsilon > 0 \, \exists \delta > 0 \text{ s.t. } \forall x, y \in [a,b] \text{ with } |x-y| < \delta, \quad |f(x) - f(y)| < \epsilon.$$

(That is, if $\delta = \delta(x_0, \epsilon)$ in the Definition above, $\inf_{x_0 \in [a,b]} \delta(x_0, \epsilon) > 0$ - in general, this infimum will be 0).

Defn. $f_n : [a,b] \to \mathbf{R}$ *converges (pointwise)* to $\delta : [a,b] \to \mathbf{R}$ if

$$f_n(x) \to f(x) \quad (n \to \infty) \quad \forall x \in [a,b].$$

Defn. $f_n : [a,b] \to \mathbf{R}$ *converges uniformly* to $f : [a,b] \to \mathbf{R}$ if

$$\forall \epsilon > 0 \, \exists N \text{ s.t. } \forall n \geq N \, \forall x \in [a,b], \quad |f_n(x) - f(x)| < \epsilon.$$

A series $\sum f_n(x)$ converges (pointwise or uniformly) if its sequence of partial sums $\sum_1^n f_k(x)$ converges (pointwise or uniformly).

We quote:

(i) Weierstrass M-test: if $|f_n(x)| \leq M_n \; \forall x \in [a,b]$ and $\sum M_n < \infty$, then $\sum f_n(x)$ converges uniformly.

(ii) If f_n are continuous, and $f_n \to f$ uniformly, then f is continuous: continuity is preserved under uniform convergence. (This does not hold in general – see Problems 1).

3. *Functions continuous on a closed interval.*

If $f : [a,b] \to \mathbf{R}$ is continuous:

1. f is *bounded*: $M = \sup_{[a,b]} f(\cdot) < \infty$, $m = \inf_{[a,b]} f(\cdot) > -\infty$.
2. f *attains its bounds*: $\exists x_1, \; x_2 \in [a,b]$ such that $f(x_1) = M$, $f(x_2) = m$.
3. *Intermediate Value Theorem*: f attains every value between its bounds: if $y \in [m, M]$, $\exists x \in [a,b]$ with $f(x) = y$.
4. *Heine's Theorem*: f is *uniformly* continous (on $[a,b]$).

Note. This is *false* if the interval is not closed. E.g., if $f(x) = 1/x$ on $(0,1]$ (f is *continuous but unbounded* on $(0,1]$, although f is bounded on $[\epsilon, 1]$ for each $\epsilon > 0$).

4. *Open and Closed Sets; Metric Spaces and Topological Spaces.*

Recall that on $\mathbf{R}$ an interval I with endpoints a, b is *open* if it omits its end-points, $I = (a,b) = \{x : a < x < b\}$, *closed* if it contains its end-points, $I = [a,b] = \{x : a \leq x \leq b\}$. Similarly for rectangles in $\mathbf{R}^2$, cuboids in $\mathbf{R}^3$, etc. What matters is the difference between openness and closedness. This is very important, can be defined quite generally, and is the basis of General Topology.