m2pm3l6(11).tex
Lecture 6. 20.1.2011.

Defn. 1. A *neighbourhood* (nhd) of a point x with radius r is $N(x,r) := \{y : |y-x| < r\}$.
2. A set S (in $\mathbf{R}^d, \mathbf{C}$,...) is *open* if each point $x \in S$ has a neighourhood in S: $\forall x \in S\, \exists r > 0$, s.t. $N(x,r) \subset S$.
3. A point x of S is a *closure point* (or *limit point*) of S if each neighbourhood $N(x)$ of x contains a point of S other then x.
Example. In $\mathbf{R}$, a, b are closure points of (a,b), but do not belong to it; and they are also closure points of $[a,b]$, and do belong to it.

We quote (the proofs are not difficult):
S is open $\Leftrightarrow$ its complement S^c is closed;
S is closed $\Leftrightarrow$ its complement S^c is open.
Defn. The *closure* $\bar{S}$ of S is $\bar{S} := S \cup \{$ closure points of $S\}$.

We quote (the proofs are not difficult):
(i) $\bar{S}$ is closed; (ii) $\bar{\bar{S}} = \bar{S}$; (iii) S closed $\Leftrightarrow S = \bar{S}$; (iv) $\bar{S}$ is the smallest closed set containing S; (v) $\bar{S}$ is the intersection of all closed sets $\supset S$.
Example. If I is $(a,b), (a,b], [a,b), [a,b]$, then $\bar{I} = [a,b]$.
Defn. $x \in S$ is an *interior point* of S if some neighbourhood $N(x) \subset S$. The set of interior points of S is S^o ("o for open"), the *interior* of S.

The following five statements are the counterparts for open sets of the five above for closed sets.
(i) S^o is open; (ii) $S^{oo} = S^{oo}$; (iii) S open $\Leftrightarrow S = S^o$; (iv) S^o is the largest open subset of S; (v) S^o is the union of all open subsets of S.
Example. If I is $(a,b), (a,b], [a,b), [a,b]$, then $I^o = (a,b)$.
Defn. The *boundary* ∂S of S is $\partial S = \bar{S} \setminus S^o$.
Example. If I is $(a,b), (a,b], [a,b), [a,b]$, then $\partial I = \{a,b\}$.
Defn. In $\mathbf{C}$, write:
$N(z_0, r) := \{z : |z - z_o| < r\}$, the open disc, centre z_0 radius r;
$\bar{N}(z_0, r) := \{z : |z - z_o| \le r\}$, the closed disc, centre z_0 radius r;
$C(z_0, r) := \{z : |z - z_o| = r\}$, the circle centre z_0, radius r.

One can check: 1. The union of an *arbitrary* family of open sets is open; 2. The intersection of a *finite* family of open sets is open.
Example. Note that in $\mathbf{R}$, $(-1/n, 1/n)$ is open $(n = 1, 2, 3, ...)$, but $\cup_{n=1}^{\infty}(-1/n, 1/n) = \{0\}$, which is closed. So the restriction to 'finite' in 2 is vital.

The two statements above for open sets have 'dual forms' for closed sets, which we obtain by taking complements (which interchanges open and closed)

and using De Morgan's laws (which interchange union and intersection):
1. The intersection of an *arbitrary* family of closed sets is closed.
2. The union of a finite family of closed sets is closed.
Note. 1. These properties of open sets form the basis of the subject of General Topology (Felix HAUSDORFF (1868-1942) in 1914), which is about *topological spaces*.
2. More general than $\mathbf{R}^d$ or $\mathbf{C}$ but less general then topological spaces are *metric spaces* (Maurice FRÉCHET (1889-1973) in 1906), spaces with a *distance function* $d = d(x, y)$ between points x, y which satisfy the *Triangle Inequality*

$$d(x, y) \leq d(x, z) + d(z, y).$$

Examples.
1. *Euclidean space* $\mathbf{R}^d$ (including $\mathbf{C}$ as $\mathbf{R}^2$): if $x = (x_1, ..., x_d)$, $y = (y_1, ..., y_d)$,

$$d(x, y) = \|x - y\| = \sqrt{\sum_{i=1}^{d} |x_i - y_i|^2}$$

(Pythagoras' Theorem: c. 450 BC).
2. *Hilbert Space* (David HILBERT (1862-1943)).
$\ell^2 := \{x = (x_n)_{n_1}^{\infty} : \sqrt{\sum_{n=1}^{\infty} |x_n|^2} < \infty\}$. Then (Pythagoras' Theorem again)

$$d(x, y) = \|x - y\| = \sqrt{\sum_{n=1}^{\infty} |x_n - y_n|^2}.$$

Hilbert space ℓ^2 (a sequence space – there are others, such as the function space L^2 ('L for Lebesgue')) can be thought of as 'Euclidean space of infinite dimension'. There are similarities between Hilbert and Euclidean spaces, but also important differences, which we shall soon meet.

5. *Infinite, countable and uncountable sets.* Write $\mathbf{N}_n = \{1, 2, ..., n\}$.
Defn (Galileo, Dedekind). A set S is *infinite* iff it can be put in 1-1 correspondence with a proper subset of itself, *finite* otherwise.
Example. The simplest infinite set (and the first we meet) is the integers $\mathbf{N}$, which is infinite under this definition because $\mathbf{N} \leftrightarrow \mathbf{N} \setminus \{1\}$ under $n \leftrightarrow n+1$.
Defn. If an infinite set S can be put in 1-1 correspondence with $\mathbf{N}$, S is called *countable*, otherwise S is called *uncountable*.
Examples. $\mathbf{N}, \mathbf{Z}, \mathbf{Q}$ are countable, $[0, 1], \mathbf{R}, \mathbf{R}^d, \mathbf{C}, \mathbf{C}^d, \ell^2$ are uncountable.