m2pm3l8(11).tex

Lecture 8. 25.1.2011.

3. The extended complex plane $\mathbf{C}^*$ is compact.

Proof. By stereographic projection, $\mathbf{C}^* \leftrightarrow \Sigma$, closed and bounded, so compact by Heine-Borel.

Theorem (Heine). If f is a continuous function on a compact set S, f is uniformly continuous on S.

Cor. (Heine). If $f : [a, b] \to \mathbf{R}$ is continuous, f is uniformly continuous on $[a, b]$.

See Handout (The theorems of Bolzano & Weierstrass, Cantor and Heine & Borel) (not examinable).

8. *Cauchy's General Principle of Convergence.*

In $\mathbf{R}^d$ or $\mathbf{C}$, a sequence (x_n) is called *Cauchy* if

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall m, n \geq N, |x_m - x_n| < \epsilon.$$

It is called *convergent* if

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall n \geq N, |x_n - x| < \epsilon.$$

Convergent $\Rightarrow$ Cauchy: For then, by the Triangle Inequality,

$$\forall m, n \geq N, |x_m - x_n| \leq |x_m - x| + |x - x_n| < \epsilon + \epsilon = 2\epsilon.$$

So (x_n) is Cauchy. //

The converse, Cauchy $\Rightarrow$ convergent, is *true* in $\mathbf{R}^d$ (or $\mathbf{C}$), but *false* in $\mathbf{Q}$ (or $\mathbf{Q}^d$). E.g. The sequence of decimal approximations to $\sqrt{2}$, $1, 1.4, 1.414, ...$ is convergent to $\sqrt{2}$ in $\mathbf{R}$, but not convergent in $\mathbf{Q}$ (as $\sqrt{2} \notin \mathbf{Q}$).

Defn. Call a metric space *complete* if all Cauchy sequences are convergent.

We quote: we can embed any metric space in a larger metric space, its *completion*, s.t.: (i) the new space is complete; (ii) the original space is *dense* in the new space (any point in the new space is a limit of a convergent sequence of points in the old space). The prototype of this is completing the rationals to get the reals (Cantor, 1872). The general procedure (which came much later, with metric spaces) is essentially no more complicated.

9. *Upper and lower limits.* The sequence $+1, -1, +1, -1, ..., (-1)^{2n}, (-1)^{2n+1}, ...$ is not convergent. But it contains convergent subsequences, and has 'upper limit $+1$, and lower limit -1'. Any sequence (x_n) of reals has an *upper and lower limit*, *limsup* and *liminf*, (possibly $\pm\infty$):

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} (\sup_{k\geq n} x_k), \qquad \liminf_{n\to\infty} x_n = \lim_{n\to\infty} (\inf_{k\geq n} x_k).$$

10. *O and o.* We write '$f(x) = O(g(x))$' (as $x \to \infty$, say) to mean '$f(x)/g(x)$ is bounded', '$f(x) = o(g(x))$' to mean '$f(x)/g(x) \to 0$. Thus $f = O(g)$ means 'f is of the order of g, while $f = o(g)$ means 'f is of smaller order than g'. We shall make heavy use of this very useful notation in Chapter III.

11. *Power Series.* Recall two convergence tests for $\sum_0^\infty a_n$.

Ratio Test (D'Alembert). If $|a_{n+1}/a_n| \to \ell$ as $n \to \infty$, $\sum a_n$ is
(i) convergent if $\ell < 1$; (ii) divergent if $\ell > 1$.
The test is inconclusive if $\ell = 1$.

Root Test (Cauchy). Write $\rho := \limsup(|a_n|^{1/n})$
(i) If $\rho < 1$, $\sum |a_n|$ is convergent ($\sum a_n$ is absolutely convergent),
(ii) If $\rho > 1$, $\sum |a_n|$ is divergent.
The test is inconclusive if $\rho = 1$.

The Root Test is more general: if $a_n > 0$, $a_{n+1}/a_n \to \ell \Rightarrow a_n^{1/n} \to \ell$ also. Usually we use the Ratio Test for a_n involving factorials $n!$, and use the Root test for a_n involving nth powers, x^n.

Defn. A *power series* in $z \in \mathbf{C}$ is a series of the form $\sum_{n=0}^\infty a_n z^n$, ($a_n \in \mathbf{C}$). The series may converge for:
all z (e.g. the exponential series $e^z = \sum_0^\infty z^n/n!$);
some but not all z (e.g. the geometric series $1/(1-z)$);
only for $z = 0$, to a_0 (e.g. $\sum_0^\infty n! z^n$) – a trivial case, omitted.