m2pm3l9.tex
Lecture 9. 27.1.2011.

Defn. A *power series* in $z \in \mathbf{C}$ is a series of the form $\sum_{n=0}^{\infty} a_n z^n$, $(a_n \in \mathbf{C})$. The series may converge for:
all z (e.g. the exponential series $e^z = \sum_0^\infty z^n/n!$);
some but not all z (e.g. the geometric series $1/(1-z)$);
only for $z = 0$, to a_0 (e.g. $\sum_0^\infty n! z^n$) – a trivial case, omitted.

We write $\rho := \limsup |a_n|^{1/n}$. By the Root Test for $\sum a_n z^n$:
If $\rho|z| < 1$, i.e. $|z| < 1/\rho$, $\sum a_n z^n$ converges (absolutely).
If $\rho|z| > 1$, i.e. $|z| > 1/\rho$, $\sum a_n z^n$ diverges.
If $\rho|z| = 1$, i.e. $|z| = 1/\rho$, the test is inconclusive.

Defn. Write $R := 1/\rho = 1/\limsup |a_n|^{1/n}$: $\sum a_n z^n$ is absolutely convergent for $|z| < R$, and divergent for $|z| > R$. We call R the *Radius of Convergence* (R of C) of $\sum a_n z^n$, $|z| = R$ its *Circle of Convergence.*
Similarly for $\sum a_n (z - z_0)^n$, with *base-point* z_0.

So a power series: converges (absolutely) *inside* its circle of convergence; diverges *outside* its circle of convergence; may do either *on* its circle of convergence. $\sum a_n z^n$ converges (absolutely and) uniformly in $|z| \le R_1$, $R_1 < R$: A power series converges (absolutely and) uniformly in *closed discs inside the circle of convergence.*

10. *Termwise differentiation and integration.*

If $\sum u_n(x)$ is a convergent series of functions and $\int \{\sum u_n(x)\,dx\} = \sum [\int u_n(x)\,dx]$, we say $\sum u_n(x)$ can be *integrated term-by-term*, or *termwise.* If $\{\sum u_n(x)\}' = \sum u_n'(x)$, we say $\sum a_n(x)$ can be *differentiated termwise.* We quote:
(I) If $\sum u_n(x)$ converges *uniformly*, it can be integrated termwise.
(D) If $\sum u_n'(x)$ *converges uniformly*, then $\sum u_n(x)$ can be differentiated termwise.

For a power series $\sum a_n z^n$, we get $\sum n a_n z^{n-1}$ by differentiating termwise; similarly we get $\sum a_n z^{n+1}/(n+1)$ by integrating termwise. All three power series have the same R of C (the shift of suffix from n to $n \pm 1$ makes no difference, and neither do the factors of n or $n+1$, as $n^{1/n} = e^{(\log n)/n} \to e^0 = 1$). Combining:

Theorem. A power series can be differentiated (or integrated) termwise inside its circle of convergence.

We can do this arbitrarily often ('infinitely often'):

Theorem. A power series can be differentiated (termwise) *infinitely often* inside its circle of convergence.

We shall see later (Cauchy-Taylor Theorem, II.7) that the functions we study in Chapter II - *holomorphic functions*, 'differentiable once', are exactly those representable by power series. So:

$$f \text{ differentiable } \textit{once} \Leftrightarrow f \text{ differentiable infinitely often.}$$

This is a total contrast to Real Analysis.

Addendum to I.1: Complex numbers (Lecture 5).
Complex nth roots of unity. For k integer, $e^{2\pi ik} = 1$. For n integer, take nth roots: $e^{2\pi ik/n} = 1$. These complex values are distinct for $k = 0, 1, \ldots, n-1$, and are called the (*complex*) *nth roots of unity.* They are on the unit circle, equally spaced at the vertices of a regular n-gon (draw a diagram to illustrate this, for $n = 2$, 3, 4, 5 and 6).

If ω is an nth root of unity, it satisfies the equation $\omega^n = 1$. Now $\omega = 1$ is one root. From the identity $\omega^n - 1 = (\omega - 1)(\omega^{n-1} + \omega^{n-2} + \cdots + \omega + 1)$, the other $n-1$ nth roots of unity satisfy

$$\omega^{n-1} + \omega^{n-2} + \cdots + \omega + 1 = 0.$$

If z is complex, $n = 1, 2, 3, \ldots$, and $z^{1/n}$ is one nth root of z, then so are $z^{1/n}\omega_n$, where ω_n runs through the n nth roots of unity. These different values (or branches) are the same when $z = 0$, which is accordingly called a *branch-point* of $z^{1/n}$. There are n nth roots: nth roots are *non-unique.* E.g., for $n = 2$ there are two square roots: even in Real Analysis, we get a sign ambiguity when we take square roots.